Example 4

Using truth tables, determine whether the following proposition is a tautology, contradiction or a contingency.

$$((p \to q) \to r) \leftrightarrow ((p \to q) \land (p \to r))$$

Example 4: tautology, contradiction or contingency?

$$\left(\overbrace{(\underbrace{(\mathbf{p} \to \mathbf{q})}_{\mathbf{A}} \to \mathbf{r})}^{\mathbf{C}} \leftrightarrow \overbrace{(\underbrace{(\mathbf{p} \to \mathbf{q})}_{\mathbf{A}} \wedge \underbrace{(\mathbf{p} \to \mathbf{r})}_{\mathbf{B}})}^{\mathbf{D}} \right)$$

p	q	r	**A** $\equiv$ $\mathbf{p} \to \mathbf{q}$	**B** $\equiv$ $\mathbf{p} \to \mathbf{r}$	**C** $\equiv$ $\mathbf{A} \to \mathbf{r}$	**D** $\equiv$ $\mathbf{A} \wedge \mathbf{B}$	**C** $\leftrightarrow$ **D**
T	T	T	**T**	**T**	**T**	**T**	**T**
T	T	F	**T**	**F**	**F**	**F**	**T**
T	F	T	**F**	**T**	**T**	**F**	**F**
T	F	F	**F**	**F**	**T**	**F**	**F**
F	T	T	**T**	**T**	**T**	**T**	**T**
F	T	F	**T**	**T**	**F**	**T**	**F**
F	F	T	**T**	**T**	**T**	**T**	**T**
F	F	F	**T**	**T**	**F**	**T**	**F**

Last column contains both T's and F's $\rightsquigarrow$ contingency

Example 5

Is it true that $((\neg(q \to p)) \land \neg r) \equiv (\neg p \lor (\neg q \lor r))$?

p	q	r	$A \equiv$ q → p	$B \equiv$ ¬ r	$C \equiv$ ¬ A	$D \equiv$ C ∧ B	$H \equiv$ ¬ p	$G \equiv$ ¬ q	$I \equiv$ G ∨ r	$J \equiv$ H ∨ I	D ≡ J
T	T	T	T	F	F	F	F	F	T	T	F
T	T	F	T	T	F	F	F	F	F	F	T
T	F	T	T	F	F	F	F	T	T	T	F
T	F	F	T	T	F	F	F	T	T	T	F
F	T	T	F	F	T	F	T	F	T	T	F
F	T	F	F	T	T	T	T	F	F	T	T
F	F	T	T	F	F	F	T	T	T	T	F
F	F	F	T	T	F	F	T	T	T	T	F

No, not true

Proof of Absorption Law

$p \vee (p \wedge q)$

$\equiv (p \vee p) \wedge (p \vee q)$ — distributive law

$\equiv p \wedge (p \vee q)$ — Idempotent

$\equiv (p \wedge p) \vee (p \wedge q)$ — distributive law

$\equiv p \vee (p \wedge q) \equiv p$ — domination law

Note: brackets avoid ambiguity!

Contrapositive, Converse, and Inverse

Definitions

The converse of $p \rightarrow q$ is $\quad q \rightarrow p$

The contrapositive of $p \rightarrow q$ is $\quad \neg q \rightarrow \neg p$

The inverse of $p \rightarrow q$ is $\quad \neg p \rightarrow \neg q$

Example: If you are a CS student, you can take COMP 238.

Converse:

If you can take COMP 238, you are a CS student.

Contrapositive:

If you cannot take COMP 238, you are not a CS student.

Inverse:

If you are not a CS student, you cannot take COMP 238.

Proof of Contrapositive

Show that

a proposition is equivalent to its contrapositive

i.e., $p \to q \equiv \neg q \to \neg p$

$$
\begin{aligned}
& p \to q \\
& \equiv \neg p \vee q && \text{(Example 1 above)} \\
& \equiv q \vee \neg p && \text{(commutative law)} \\
& \equiv \neg(\neg q) \vee \neg p && \text{(double negation law)} \\
& \equiv \neg q \to \neg p && \text{(Example 1 above)}
\end{aligned}
$$

However, neither the inverse nor the converse of $p \to q$ is logically equivalent to it.

Example 6

Using equivalences, simplify

$$((q \wedge r) \wedge (q \leftrightarrow p)) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\Big((q \wedge r) \wedge \underbrace{(q \leftrightarrow p)}_{(p \wedge q) \vee (\neg p \wedge \neg q)}\Big) \rightarrow \neg(r \rightarrow \neg p)$$

according to what we proved in Slide 42

$$\equiv (q \wedge r) \wedge \Big((q \wedge p) \vee (\neg q \wedge \neg p)\Big) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Alternate Solution

$$\Big((q \wedge r) \wedge \underbrace{(q \leftrightarrow p)}_{(p \to q) \wedge (p \to q)}\Big) \to \neg(r \to \neg p$$

$$\equiv (q \wedge r) \wedge \big(\underbrace{(p \to q)}_{\neg p \vee q} \wedge \underbrace{(q \to p)}_{\neg q \vee p}\big)$$

$$\to \neg(r \to \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv (q \wedge r) \wedge \Big((q \wedge p) \vee (\neg q \wedge \neg p) \Big) \rightarrow \neg (r \rightarrow \neg p)$$

Distribution of $\wedge$

$$\equiv \Bigg(\Big((q \wedge r) \wedge (q \wedge p) \Big) \vee \Big((q \wedge r) \wedge ((\neg q) \wedge (\neg p)) \Big) \Bigg) \rightarrow \neg (r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \Bigg(\Big((q \wedge r) \wedge (q \wedge p) \Big) \vee \Big((q \wedge r) \wedge ((\neg q) \wedge (\neg p)) \Big) \Bigg)$$

$$\rightarrow \neg (r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \Big(\big((q \wedge r) \wedge (q \wedge p) \big) \vee \big((q \wedge r) \wedge ((\neg q) \wedge (\neg p)) \big) \Big) \rightarrow \neg(r \rightarrow \neg p)$$

Remove unnecessary parenthesis and simplify

$$\equiv \Big(\big(q \wedge r \wedge p \big) \vee \big(q \wedge (\neg q) \wedge r \wedge (\neg p) \big) \Big) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left(q \wedge r \wedge p \right) \vee \left(\overbrace{q \wedge (\neg q)}^{\mathbf{F}} \wedge r \wedge (\neg p) \right) \right) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left(q \wedge r \wedge p \right) \vee \left(\overbrace{q \wedge (\neg q)}^{\mathbf{F}} \wedge r \wedge (\neg p) \right) \right) \to \neg(r \to \neg p)$$

$$\equiv \left(\left((q \wedge r \wedge p \right) \vee \left(\mathbf{F} \wedge r \wedge (\neg p) \right) \right) \to \neg(r \to \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left((q \wedge r \wedge p \right) \vee \left(\overbrace{\mathbf{F} \wedge r \wedge (\neg p)}^{\mathbf{F}} \right) \right)$$

$$\rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left((q \wedge r \wedge p) \right) \vee \left(\overbrace{\mathbf{F} \wedge r \wedge (\neg p)}^{\mathbf{F}} \right) \right) \rightarrow \neg(r \rightarrow \neg p)$$

$$\equiv \left(\left((q \wedge r \wedge p) \right) \vee \mathbf{F} \right) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left((q \wedge r \wedge p\right) \vee \mathbf{F}\right) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(\left((q \wedge r \wedge p) \right) \vee \mathbf{F} \right) \rightarrow \neg(r \rightarrow \neg p)$$

Simplification: Elimination of **F**

$$\equiv \left(q \wedge r \wedge p \right) \rightarrow \neg(r \rightarrow \neg p)$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \Big(q \;\wedge\; r \;\wedge\; p\Big) \quad \rightarrow \quad \neg(\mathbf{r} \rightarrow \neg\mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \Big(q \;\wedge\; r \;\wedge\; p\Big) \quad \rightarrow \quad \neg(\mathbf{r} \rightarrow \neg\mathbf{p})$$

Implication: $p \rightarrow q \equiv \neg p \vee q$

$$\equiv \quad \Big(q \;\wedge\; r \;\wedge\; p\Big) \quad \rightarrow \quad \neg(\neg\mathbf{r} \vee \neg\mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \Big(q \;\wedge\; r \;\wedge\; p\Big) \quad \rightarrow \quad \neg(\neg \mathbf{r} \vee \neg \mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \left(q \wedge r \wedge p \right) \rightarrow \neg(\neg \mathbf{r} \vee \neg \mathbf{p})$$

De Morgan

$$\equiv \left(q \wedge r \wedge p \right) \rightarrow \mathbf{r} \wedge \mathbf{p}$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \left(q \quad \wedge \quad r \quad \wedge \quad p\right) \quad \rightarrow \quad \mathbf{r} \wedge \mathbf{p}$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \Big(q \wedge r \wedge p\Big) \rightarrow \mathbf{r} \wedge \mathbf{p}$$

Implication: $p \rightarrow q \equiv \neg p \vee q$

$$\equiv \quad \neg\Big(q \wedge r \wedge p\Big) \vee (\mathbf{r} \wedge \mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \neg\Big(q \wedge r \wedge p\Big) \vee (\mathbf{r} \wedge \mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \neg\left(q \wedge r \wedge p\right) \vee (\mathbf{r} \wedge \mathbf{p})$$

De Morgan

$$\equiv \neg\mathbf{q} \vee \neg(r \wedge p) \vee (\mathbf{r} \wedge \mathbf{p})$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \neg q \quad \vee \quad \neg(\mathbf{r} \wedge \mathbf{p}) \quad \vee \quad (\mathbf{r} \wedge \mathbf{p})$$

Simplification

$$\equiv \quad \neg q \vee \mathbf{T}$$

$$\equiv \quad \mathbf{T}$$

Example 6: Simplification using equivalencies - Cont'd

$$\equiv \quad \neg q \quad \vee \quad \neg(\mathbf{r} \wedge \mathbf{p}) \quad \vee \quad (\mathbf{r} \wedge \mathbf{p})$$

Simplification

$$\equiv \quad \neg q \vee \mathbf{T}$$

$$\equiv \quad \mathbf{T}$$

Conclusion: back to Slide 51

$$((q \wedge r) \wedge (q \leftrightarrow p)) \rightarrow \neg(r \rightarrow \neg p) \equiv T.$$