ON THE COMPUTATIONAL COMPLEXITY OF FINDING A KERNEL

by

V. CHVÁTAL

June 1973

Centre de recherches mathématiques, Université de Montréal

CRM - 300

The purpose of this note is to exhibit a new member of Karp's class of "complete" problems. For the notation and necessary background, the reader is referred to Karp's paper [3].

THEOREM. The following problem is complete.

KERNEL

INPUT: digraph H

PROPERTY: there is a set $K \subset V$ such that

(i) $\langle u,v \rangle \in E$ for no $u,v \in K$

(ii) if $u \notin K$ then there is a $v \in K$ with $\langle u,v \rangle \in E$.

Proof. We will show that SATISFIABILITY $\propto$ KERNEL. The reduction (still in Karp's notation) goes as follows.

$$V = V_1 \cup V_2 \ , \ E = E_1 \cup E_2 \cup E_3 \cup E_4 \ ,$$

$$V_1 = \{\langle \sigma,i \rangle \mid \sigma \text{ is a literal and occurs in } C_i\} \ ,$$

$$V_2 = \{1,2,\ldots,p\} \times \{1,2,3\} \ ,$$

$$E_1 = \{\langle\langle \sigma,i \rangle , \langle \delta,j \rangle\rangle \mid i = j \ , \sigma \neq \delta\} \ ,$$

$$E_2 = \{\langle\langle \sigma,i \rangle , \langle \delta,j \rangle\rangle \mid i \neq j \ , \sigma = \overline{\delta}\} \ ,$$

$$E_3 = \{\langle\langle i,k \rangle , \langle \sigma,i \rangle\rangle \mid \langle i,k \rangle \in V_2 \ , \langle \sigma,i \rangle \in V_1\} \ ,$$

$$E_4 = \bigcup_{i=1}^{p} \{\langle\langle i,1 \rangle , \langle i,2 \rangle\rangle , \langle\langle i,2 \rangle , \langle i,3 \rangle\rangle , \langle\langle i,3 \rangle , \langle i,1 \rangle\rangle\} \ .$$

REMARK. The problem of characterizing digraphs that have a kernel, resp. the problem of finding a kernel in a given digraph attracted attention of several authors [1], [2], [4].

REFERENCES

[1] C. BERGE, Graphs and Hypergraphs, North Holland, Amsterdam 1973, Chapter 14. Kernels and Grundy functions.

[2] F. HARARY and M. RICHARDSON, A matrix algorithm for solutions and r-bases of a finite irreflexive relation, Naval Logistics Quart. VI (1959), 307-314.

[3] R.M. KARP, Reducibility among combinatorial problems, Complexity of computer computations (R.E. Miller et al., eds.), Plenum Press, 1972.

[4] J. VON NEUMANN and O. MORGENSTERN, Theory of Games and Economic Behavior, Princeton University Press, 1944.