COMP361 NUMERICAL METHODS TERM TEST 1

NAME:	ID:

INSTRUCTIONS

You may only use a Concordia approved calculator.

Other electronic equipment, books and notes, are not allowed.

This exam consists of 8 Problems, worth 8 points each.

The total number of pages is 9. Check that your copy is complete !

Do not attach or detach any pages !

Rough work can be done in the open spaces and on the back of the pages.

Use a *dark black* or *dark blue* pen to enter your final answer.

For multiple choice questions, circle the correct answer (only one answer is correct).

Where provided, put you final answer in the box(es).

Table for Instructor's use

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7	Problem 8	TOTAL

Problem 1.

For all vectors $\mathbf{x}$ in $\mathbb{R}^n$ we have

$\| \mathbf{x} \|_2 \quad \leq \quad \| \mathbf{x} \|_1$ True False

TRUE !

For all vectors $\mathbf{x}$ in $\mathbb{R}^n$ we have

$\| \mathbf{x} \|_2 \quad \leq \quad \| \mathbf{x} \|_\infty$ True False

FALSE !

For all n by n matrices $\mathbf{A}$ we have

$\| \mathbf{A} \|_1 \quad \leq \quad \| \mathbf{A} \|_\infty$ True False

FALSE !

For all n by n matrices $\mathbf{A}$ we have

$\| \mathbf{A} \|_\infty \quad \leq \quad \| \mathbf{A} \|_1$ True False

FALSE !

Problem 2.

For all square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension n we have

$\| \mathbf{A} + \mathbf{B} \|_\infty \leq \| \mathbf{A} \|_\infty + \| \mathbf{B} \|_\infty$ True False

TRUE !

For all square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension n we have

$\| \mathbf{AB} \|_\infty \leq \| \mathbf{A} \|_\infty \ \| \mathbf{B} \|_\infty$ True False

TRUE !

For all square, invertible matrices $\mathbf{A}$ of dimension n we have

$\mathrm{cond}(\mathbf{A}) \geq 1$ True False

TRUE !

For all square, invertible matrices $\mathbf{A}$ of dimension n we have

$\mathrm{cond}(\mathbf{A}^{-1}) = \mathrm{cond}(\mathbf{A})$ True False

TRUE !

Problem 3.

What is the number of operations needed for solving a general n by n *tridiagonal system* of linear equations? (Here an "operation" is a multiplication or a division.)

1. $3n - 3$
2. $4n - 4$
3. $5n - 4$ THIS ONE !
4. none of the above

Suppose that solving a general linear system of equations of dimension n requires 0.1 second on a given computer when $n = 10^2$. Based on the number of operations (multiplications and divisions only) estimate how much time it will take to solve a general linear system of equations of dimension $n = 10^3$ on this computer.

1. 1 second.
2. 10 seconds.
3. 100 seconds. THIS ONE !
4. 1000 seconds.

For general n by n matrices $\mathbf{A}$ and $\mathbf{B}$:

The matrix product $\mathbf{AB}$ can be computed using n^3 multiplications. True False

TRUE !

For each of the following three matrices put their *condition number* in the box below it:

(1) $\begin{pmatrix} 10^9 & 0 \\ 0 & 10^9 \end{pmatrix}$ (2) $\begin{pmatrix} 10^9 & 0 \\ 0 & 10^{-9} \end{pmatrix}$ (3) $\begin{pmatrix} 10^{-9} & 0 \\ 0 & 10^{-9} \end{pmatrix}$

(1)	(2)	(3)
1.0	10^{18}	1.0

Problem 4. Consider the 3 by 3 matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} .$$

In the space below, make use of the *Banach Lemma* to show that $\mathbf{A}$ is invertible.

Also make use of the Banach Lemma to determine an upper bound on $\| \mathbf{A}^{-1} \|_\infty$ and an upper bound on the condition number of $\mathbf{A}$, using the matrix infinity-norm. Enter your answers in the appropriate box below:

$\| \mathbf{A}^{-1} \|_\infty \le 1$	$\mathrm{cond}(\mathbf{A}) \le 5$

Problem 5.

Consider the 3 by 3 matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} .$$

Carry out the following computations *in the given order*, and enter your answers in the appropriate box below. (You may use exact, rational arithmetic.)

(a) Use Gauss elimination to compute the **LU**-decomposition of the 3 by 3 matrix **A**.

Having computed **L** and **U**, let $\mathbf{f} = (3, 0, 5)^T$, and carry out the following computations:

(b) Solve $\mathbf{Lg} = \mathbf{f}$ for $\mathbf{g}$.

(c) Solve $\mathbf{Ux} = \mathbf{g}$ for $\mathbf{x}$ (and check that your answer is correct !)

$$\boxed{\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix}} \qquad \boxed{\mathbf{U} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{pmatrix}}$$

$$\boxed{\mathbf{g} = \begin{pmatrix} 3 \\ -\frac{3}{2} \\ 4 \end{pmatrix}} \qquad \boxed{\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}$$

Problem 6

Write down Newton's method for computing *the cube root of* 5 in this box:

$$\boxed{}$$

Carry out the first two iterations of Newton's Method with $x^{(0)} = 2.0$:

$$\boxed{\begin{aligned} x^{(1)} &= \tfrac{7}{4} = 1.75 \\ x^{(2)} &\approx 1.7108844 \end{aligned}}$$

Also write down the Chord method for *the cube root of* 5 in this box:

$$\boxed{}$$

Carry out the first two iterations of the Chord method, with $x^{(0)} = 2.0$.

$$\boxed{\begin{aligned} x^{(1)} &= \tfrac{7}{4} = 1.75 \\ x^{(2)} &\approx 1.7200521 \end{aligned}}$$

Problem 7.

Consider the the *nonlinear system of equations*

$$x_1^4 + x_2^4 - 1 = 0 \,,$$

$$x_1^2 - x_2 = 0 \,.$$

Carry out *the first iteration* of Newton's method for solving the above system, using $x_1^{(0)} = 1$, and $x_2^{(0)} = 0$, as initial guesses. Show details of your computations, and enter the values of $x_1^{(1)}$ and $x_2^{(1)}$ in the box below.

NOTE : This problem is meant to illustrate Newton's Method for solving a system of equations.

Thus do not reduce the two equations to a single equation.

$x_1^{(1)} = 1.0$

$x_2^{(1)} = 1.0$

Problem 8.

Consider the fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where

$$f(x) = c\,x(1-x)\ , \qquad \text{with} \qquad c = \frac{7}{2}\ .$$

(a) Analytically determine all fixed points of this fixed point iteration, and for each fixed point determine whether it is attracting or repelling. If a fixed point is attracting then also determine if the convergence is linear or quadratic. Enter your final answers in the box below.

$x = 0$	repelling
$x = \frac{5}{7}$	repelling

(b) In the space below, draw the standard graphical interpretation of this fixed point iteration, showing the line $y = x$, the curve $y = f(x)$, and indicating several iterations, starting with $x^{(0)} \cong 0.1$. Make sure that your graph is qualitatively accurate.

SEE PAGES 118-125 OF THE LECTURE NOTES FOR SIMILAR EXAMPLES