COMP361 NUMERICAL METHODS TERM TEST 2

(with abbreviated solutions)

NAME:	ID:

INSTRUCTIONS

You may only use a Concordia approved calculator.

Other electronic equipment, books and notes, are not allowed.

This exam consists of 8 Problems, worth 8 points each.

The total number of pages is 9. Check that your copy is complete !

Do not attach or detach any pages !

Rough work must be done on the back of the pages.

Use a *dark black* or *dark blue* pen to enter your final answer.

For multiple choice questions, circle the correct answer (only one answer is correct).

Where provided, put you final answer(s) in the box(es).

Table for Instructor's use

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7	Problem 8	TOTAL

Problem 1.

For all vectors $\mathbf{x}$ in $\mathbb{R}^n$ we have

(a) $\| \mathbf{x} \|_1 \leq \| \mathbf{x} \|_2$ (b) $\| \mathbf{x} \|_\infty \leq \| \mathbf{x} \|_2$

1. Both (a) and (b) are valid.
2. (a) is valid and (b) is invalid.
3. (a) is invalid and (b) is valid. THIS ONE !
4. Both (a) and (b) are invalid.

Suppose that *multiplying* two general square matrices of dimension n requires 1 second on a given computer when $n = 10^2$. Based on the number of operations (multiplications and divisions only) estimate how much time it will take to *solve a general linear system* of equations of dimension $n = 10^3$ on this computer.

1. 33 seconds.
2. 333 seconds. THIS ONE !
3. 3333 seconds.
4. none of the above

Let $\mathbf{T}_n$ be the n by n *tridiagonal matrix* $\mathbf{T}_n = \text{diag}[1, 5, 1]$. Which one of the following upper bounds on $\| \mathbf{T}_n^{-1} \|_\infty$ is obtained using the Banach Lemma:

1. $\frac{1}{3}$ THIS ONE !
2. $\frac{2}{3}$
3. 1
4. none of the above

Again let $\mathbf{T}_n$ be the specific n by n *tridiagonal matrix* $\mathbf{T}_n = \text{diag}[1, 5, 1]$. What is the total of the number of multiplications and divisions needed to determine the **LU**-decomposition of $\mathbf{T}_n$, when taking into account its tridiagonal structure, as well as the specific entries that it contains. (Make sure not to include unnecessary operations !) :

1. $n - 1$ THIS ONE !
2. $2n - 1$
3. $3n - 2$
4. none of the above

Problem 2.

Only one of the following statements is valid. Circle the one that is valid.

1. All very large matrices are ill-conditioned.

2. All ill-conditioned matrices have very small determinants.

3. If pivoting is needed then the matrix is ill-conditioned.

4. The condition number of a matrix is never smaller than 1. THIS ONE !

Newton's method for computing the square root of 2 converges to *the positive square root of* 2 for the following initial guesses $x^{(0)}$:

1. all $x^{(0)}$
2. all $x^{(0)} \neq 0$
3. all $x^{(0)} > 0$ THIS ONE ! (See Page 135 of the Lecture Notes)
4. none of the above

The fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where $f(x) = \cos(x)$, has

1. no attracting fixed point
2. one attracting fixed point with linear convergence THIS ONE !
3. one attracting fixed point with quadratic convergence
4. none of the above

The fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where $f(x) = \sin(x)$, has

1. no attracting fixed point
2. one attracting fixed point with linear convergence
3. one attracting fixed point with quadratic convergence
4. none of the above THIS ONE !

Problem 3

The fixed point iteration $x^{(k+1)} = f(x^{(k)})$, where $f(x) = x^2 - 2x + 2$, has

1. only one attracting fixed point, with linear convergence
2. only one attracting fixed point, with quadratic convergence THIS ONE !
3. more than one attracting fixed points
4. none of the above

Consider *Newton's method for systems* for solving the system of equations

$$x_1^2 + x_2^2 - 1 = 0 ,$$

$$x_1^3 - x_2 = 0 .$$

If one uses $x_1^{(0)} = 1$, and $x_2^{(0)} = 0$, as initial guesses, then what will be the values of $x_1^{(1)}$ and $x_2^{(1)}$:

1. $x_1^{(1)} = 1$, and $x_2^{(1)} = \frac{1}{2}$
2. $x_1^{(1)} = \frac{1}{2}$, and $x_2^{(1)} = 1$
3. $x_1^{(1)} = 1$, and $x_2^{(1)} = 1$ THIS ONE !
4. none of the above

Which of the following polynomials represents *the first two nonzero terms* of the Taylor polynomial for $f(x) = \sin(x)$ about the point $x = 0$.

1. $x - \frac{x^2}{2}$
2. $x - \frac{x^3}{6}$ THIS ONE !
3. $x + \frac{x^3}{6}$
4. none of the above.

For the Taylor polynomial $p_n(x)$ of degree n (or less) for $f(x) = e^x$ about the point $x = 0$, what is the smallest value of n so that $| e^x - p_n(x) | < 10^{-2}$ everywhere in the interval $[-1, 1]$:

1. $n = 3$
2. $n = 4$
3. $n = 5$ THIS ONE !
4. none of the above.

Problem 4.

If we approximate $f(x) = e^x$ in the interval $[-1, 1]$ by interpolation at $n + 1$ Chebyshev points with a polynomial of degree n or less, then *what is the smallest value of* n so that the maximum interpolation error is less than 10^{-2} ? (Make use of the Lagrange Interpolation Theorem.)

1. $n = 3$
2. $n = 4$ THIS ONE !
3. $n = 5$
4. none of the above

Suppose we approximate $f(x) = e^x$ in the interval $[-1, 1]$ by *local polynomial interpolation* at three Chebyshev points with *local polynomials* of degree 2 or less. From the choices below, what is *the smallest number of intervals of equal size* needed so that the maximum interpolation error is less than 10^{-3} :

1. $N = 5$ THIS ONE !
2. $N = 7$
3. $N = 9$
4. none of the above

For small positive h, the finite difference expression

$$\frac{f(2h) - 2f(h) + f(0)}{h^2} ,$$

can be used to approximate the following derivative:

1. $f'(0)$
2. $f''(0)$ THIS ONE !
3. $f'''(0)$
4. none of the above

How many *function evaluations* are needed when using the *composite Simpson Rule* to integrate a function $f(x)$ over an interval $[a, b]$, based on *local integration* in N subintervals of size $h = (b-a)/N$.

1. $N + 1$
2. $2N + 1$ THIS ONE !
3. $3N$
4. none of the above

Problem 5.

Consider the fixed point iteration $x^{(k+1)} \;=\; f(x^{(k)})$, where

$$f(x) \;=\; \frac{2x^3+2}{3x^2} \;.$$

(a) Analytically determine all fixed points of this fixed point iteration, and for each fixed point determine whether it is attracting or repelling. If a fixed point is attracting then also determine if the convergence is linear or quadratic. Show the details of your computations, and enter your final answers in the box below.

This is Newton for $x^3 - 2 = 0$.

The only fixed point is $x^* = 2^{\frac{1}{3}}$.

Convergence is quadratic.

(b) In the space below, draw the standard graphical interpretation of this fixed point iteration, showing the line $y = x$, the curve $y = f(x)$, and indicating several iterations, starting with $x^{(0)} \cong 0.5$. Make sure that your graph is qualitatively accurate.

Problem 6.

(a) Write down the Lagrange interpolating polynomial $p(x)$ that interpolates the function $f(x) = e^x$ at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. You must write $p(x)$ using *the complete expressions* for the appropriate Lagrange basis functions $\ell_0(x)$, $\ell_1(x)$, and $\ell_2(x)$. Enter your answer in the box.

$$p(x) \;=\; e^{x_0}\,\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \;+\; e^{x_1}\,\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \;+\; e^{x_2}\,\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

where $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$

(b) *Use the interpolating polynomial* $p(x)$ determined above to approximate the value of $f(x) = e^x$ at the point $x = 0.5$. Enter this value in the box below.

$e^{0.5} \;\approx\; 1.72337$

(exact value is $1.6487 \cdots$)

Problem 7.

(a) Use Taylor expansions to determine the leading error term of the numerical differentiation formula

$$f''(0) \;\cong\; \frac{f(2h) - 2f(h) + f(0)}{h^2} \; .$$

Show the details of your work, and enter your final answer in the box below.

NOTE: This is a so-called "one-sided approximation", as opposed to a "central approximation".

The leading error term is $h\ f'''(0)$

(b) Showing all details, derive the numerical differentiation formula in Part (a) of this Problem. You must make use of the Lagrange interpolation polynomial that interpolates $f(x)$ at the points, $x_0 = 0$, $x_1 = h$, and $x_2 = 2h$, and the corresponding Lagrange basis functions, $\ell_0(x)$, $\ell_1(x)$, and $\ell_2(x)$.

The derivation is similar (but not identical) to the derivation on Page 234 of the Lecture Notes:

Letting $x_0 = 0$, $x_1 = h$, and $x_2 = 2h$, we have

$$f''(x_0) \;\cong\; f_0\ \ell_0''(x_0) \;+\; f_1\ \ell_1''(x_0) \;+\; f_2\ \ell_2''(x_0) \; .$$

Here

$$l_0(x) \;=\; \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \; ,$$

so that

$$\ell_0''(x) \;=\; \frac{2}{(x_0 - x_1)(x_0 - x_2)} \;=\; \frac{1}{h^2} \; .$$

In particular,

$$\ell_0''(x_0) \;=\; \frac{1}{h^2} \; .$$

Similarly

$$\ell_1''(x_0) \;=\; -\frac{2}{h^2} \; , \qquad \ell_2''(x_0) \;=\; \frac{1}{h^2} \; .$$

Hence

$$f''(x_0) \;\cong\; \frac{f_0 - 2f_1 + f_2}{h^2} \; .$$

Problem 8.

The formula

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f(x)\ dx \ \approx\ h\ f(0)\ ,$$

defines the *local Midpoint Rule* for the reference interval $[-\frac{h}{2}, \frac{h}{2}]$.

(a) Use Taylor expansions to derive the *leading error term* for this local formula. Show the details of your work and enter your final answer in the box below.

See the page following Page 285 in the Lecture Notes with Solutions

(b) Based on the leading error term in the local formula, determine a bound on the leading error term of the *composite Midpoint Rule*, when integrating a function $f(x)$ over an interval $[a, b]$. Enter your answer in the box.

See the page following Page 285 in the Lecture Notes with Solutions

(c) Determine how many subintervals of equal size are needed to numerically integrate the function $f(x) = e^x$ over the interval $[-1, 1]$ using the *composite Midpoint Rule*, so that the error is less than 10^{-4} . Show the details of your work and enter your final answer in the box below.

$N \geq 96$