Gauss Elimination and LU-decomposition

(before Page 25 of the Lecture Notes)

Elimination:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 4 & -1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}}_{\mathbf{f}}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & -2 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{1}{2} & 1 \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -\frac{15}{2} \end{pmatrix}}_{\mathbf{U}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 6 \\ -9 \\ -\frac{15}{2} \end{pmatrix}}_{\mathbf{g}}$$

From the preceding page:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{1}{2} & 1 \end{pmatrix} , \qquad \overset{\mathbf{U}}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -\frac{15}{2} \end{pmatrix}} \overset{\mathbf{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \overset{=}{=} \overset{\mathbf{g}}{\begin{pmatrix} 6 \\ -9 \\ -\frac{15}{2} \end{pmatrix}}$$

Backsubstitution:

$$x_3 = (-\tfrac{15}{2})/(-\tfrac{15}{2}) = 1 ,$$

$$x_2 = \frac{-9 - (-5)(1)}{-4} = 1 ,$$

$$x_1 = \frac{6 - (3)(1) - (2)(1)}{1} = 1 .$$

NOTE: A computer uses REAL or DOUBLE PRECISION arithmetic !

NOTE:

By construction we have that

$$\begin{array}{ccc} \mathbf{L} & \mathbf{U} & = & \mathbf{A} \\ \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{1}{2} & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -\frac{15}{2} \end{pmatrix} & = & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 4 & -1 \end{pmatrix} \end{array}$$

Thus we have *decomposed* $\mathbf{A}$ into $\mathbf{L}$ and $\mathbf{U}$,

where

$\mathbf{L}$ is *lower-triangular* , with 1's along the main diagonal ,

and

$\mathbf{U}$ is *upper-triangular* .