ASSIGNMENT PROBLEMS

for

ELEMENTARY NUMERICAL METHODS

with Solutions

PROBLEM 1 :

Let $\mathbf{x} \in \mathbb{R}^3$ be given by $\mathbf{x} = (3, 4, -5)^T$.

What is the value of $\| \mathbf{x} \|_2$?

SOLUTION : $\sqrt{50}$.

PROBLEM 2 :

Use the Banach Lemma (with the matrix infinity norm) to prove that the n by n matrix $\mathbf{T}_n$ given below is invertible for all positive integers n :

$$\mathbf{T}_n \;=\; \text{diag}[1,1,5,1,1] \;\equiv\; \begin{pmatrix} 5 & 1 & 1 & & & & \\ 1 & 5 & 1 & 1 & & & \\ 1 & 1 & 5 & 1 & 1 & & \\ & 1 & 1 & 5 & 1 & 1 & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & 1 & 5 & 1 \\ & & & & 1 & 1 & 5 \end{pmatrix} .$$

Hint: First rewrite $\mathbf{T}_n$ as $\mathbf{T}_n = c\,\tilde{\mathbf{T}}_n$, where c is a constant, chosen so that the Banach Lemma can be applied to $\tilde{\mathbf{T}}_n$.

Also use the Banach Lemma to derive an upper bound on the infinity norm of the inverse matrix $\mathbf{T}_n^{-1}$, and on the condition number of $\mathbf{T}_n$.

SOLUTION :

Take $c = 5$ (the number along the main diagonal of $\mathbf{T}_n$).

Then

$$\mathbf{T}_n = 5\ (\mathbf{I}_n + \mathbf{B}_n)\ ,$$

where $\mathbf{I}_n$ is the n by n identity matrix, and

$$\mathbf{B}_n \ = \ \text{diag}[\ \frac{1}{5}\ ,\ \frac{1}{5}\ ,\ 0,\ \frac{1}{5}\ ,\ \frac{1}{5}\ ] \ \equiv \ \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} & & & & \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & & & \\ \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & & \\ & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ & & & & \frac{1}{5} & \frac{1}{5} & 0 \end{pmatrix} ,$$

Here $\| \mathbf{B}_n \|_\infty \ = \ \frac{4}{5} \ < \ 1$, so $\mathbf{I}_n + \mathbf{B}_n$ is invertible, and

$$\| \ (\mathbf{I}_n + \mathbf{B}_n)^{-1} \ \|_\infty \ \leq \ \frac{1}{1 - \| \ \mathbf{B}_n \ \|_\infty} \ = \ \frac{1}{1 - \frac{4}{5}} \ = \ 5\ .$$

From

$$\mathbf{T}_n = 5 \ (\mathbf{I}_n + \mathbf{B}_n) \ ,$$

it also follows that

$$\mathbf{T}_n^{-1} = \frac{1}{5} \ (\mathbf{I}_n + \mathbf{B}_n)^{-1} \ ,$$

so that

$$\| \ \mathbf{T}_n^{-1} \ \|_\infty \ = \ \frac{1}{5} \ \| \ (\mathbf{I}_n + \mathbf{B}_n)^{-1} \ \|_\infty \ \leq \ \frac{1}{5} \ \cdot \ 5 \ = \ 1 \ ,$$

and

$$\text{cond}(\mathbf{T}_n) \ \equiv \ \| \ \mathbf{T}_n \ \|_\infty \ \ \| \ \mathbf{T}_n^{-1} \ \|_\infty \ \leq \ 9 \ \cdot \ 1 \ = \ 9 \ .$$

PROBLEM 3 :

Use the Banach Lemma (with the matrix infinity norm) to prove that the n by n matrix $\mathbf{S}_n$ given below is invertible for all positive integers n :

$$\mathbf{S}_n \;=\; \mathrm{diag}[h_i, 2(h_i + h_{i+1}), h_{i+1}] \;\equiv$$

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & & \\ & h_2 & 2(h_2+h_3) & h_3 & & & \\ & & h_3 & 2(h_3+h_4) & h_4 & & \\ & & \cdot & \cdot & \cdot & & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & h_{n-1} & 2(h_{n-1}+h_n) \end{pmatrix},$$

where $h_i > 0$, for all i. (This matrix arises in cubic spline interpolation.)

Hint: First rewrite $\mathbf{S}_n$ as $\mathbf{S}_n = \mathbf{D}_n\ \tilde{\mathbf{S}}_n$, where $\mathbf{D}_n$ is a diagonal matrix, chosen so that the Banach Lemma can be applied to $\tilde{\mathbf{S}}_n$.

SOLUTION :

The solution of this problem is actually given on the page that follows Page 68 of the "slides-with-solutions".

PROBLEM 4 :

Use Gauss elimination to compute (pencil and paper) the **LU**-decomposition of the matrix

$$\mathbf{A} \;=\; \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \;.$$

After having computed **L** and **U**, use them to solve for $\mathbf{x}$ in

$$\mathbf{A}\mathbf{x} = \mathbf{f} \;,$$

where $\mathbf{f} = (2, 2, 1)^T$.

Check your answer !

SOLUTION :

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} , \qquad \mathbf{U} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{pmatrix} .$$

From $\mathbf{Lg} = \mathbf{f}$ we get

$$\mathbf{g} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} ,$$

and backsubstitution in $\mathbf{Ux} = \mathbf{g}$ gives

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} .$$

The purpose of this exercise is to understand use of the LU-decomposition algorithm.

PROBLEM 5 :

Give an example of a *singular* 2×2 matrix $\mathbf{A}$:

that *has* an **LU**-decomposition.

Show **L** and **U**.

SOLUTION :

One of infinitely many possible examples of a *singular* 2×2 matrix $\mathbf{A}$ that *has* an **LU**-decomposition:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} ,$$

for which

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \qquad \mathbf{U} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} .$$

PROBLEM 6 :

Give an example of a *nonsingular* 2×2 matrix $\mathbf{A}$:

that does *not have* an **LU**-decomposition.

Explain why not.

SOLUTION :

One of infinitely many possible examples of a *nonsingular* 2×2 matrix $\mathbf{A}$ that does *not have* an **LU**-decomposition:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The reason is that a *zero-division* occurs right at the first elimination step.

This problem can be fixed by using *pivoting*, *i.e.*, interchanging rows.

PROBLEM 7 :

A square matrix $\mathbf{A}$ is said to be *ill-conditioned* if the *condition number*

$$\text{cond}(\mathbf{A}) \equiv \| \mathbf{A} \| \ \| \mathbf{A}^{-1} \| ,$$

is large.

Give an example of a *nonsingular* 2×2 matrix $\mathbf{A}$ that is *ill-conditioned*, with

$$\text{cond}(\mathbf{A}) \geq 10^6 ,$$

but where the multiplier that arises in the **LU**-decomposition is not big in absolute value.

SOLUTION :

An example of a *nonsingular* 2×2 matrix $\mathbf{A}$ that is *ill-conditioned*, where the multiplier that arises in the **LU**-decomposition is not big in absolute value:

$$\mathbf{A} \ = \ \begin{pmatrix} 10^4 & 0 \\ 0 & 10^{-4} \end{pmatrix} \ ,$$

with inverse

$$\mathbf{A}^{-1} \ = \ \begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^4 \end{pmatrix} \ .$$

Here

$$\text{cond}(\mathbf{A}) \ \equiv \ \| \, \mathbf{A} \, \|_\infty \ \ \| \, \mathbf{A}^{-1} \, \|_\infty \ = \ 10^8 \ ,$$

and the sole multiplier that arises is zero !

PROBLEM 8 :

Give an example of a 2×2 matrix $\mathbf{A}$ that is *not ill-conditioned*, with

$$\text{cond}(\mathbf{A}) \ \leq \ 10 \ ,$$

but where the multiplier that arises in the $\mathbf{LU}$ -decomposition is of very large magnitude, *i.e.*, is big in absolute value.

SOLUTION :

An example of a 2×2 matrix $\mathbf{A}$ that is *not ill-conditioned*, where the multiplier that arises in the **LU**-decomposition is of large magnitude:

Consider the matrix

$$\mathbf{A} \ = \ \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

which has inverse

$$\mathbf{A}^{-1} \ = \ \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \ .$$

Take c to be small, for example, $c = 10^{-9}$. Then

$$\text{cond}(\mathbf{A}) \ \equiv \ \| \mathbf{A} \| \ \ \| \mathbf{A}^{-1} \| \ = (c+1)(1+c) \ = \ (1+c)^2 \ \approx \ 1 \ ,$$

whereas the sole multiplier that arises is $1/c \ = \ 10^9$.

PROBLEM 9 :

Compute (pencil and paper) the **LU**-decomposition of the $n \times n$ *Hilbert matrix* $\mathbf{H}_n$ whose entries in the ith row and jth column are

$$h_{i,j} \;=\; 1/(i+j-1), \quad i,j = 1,\cdots,n \ ,$$

for the case $n \;=\; 2$.

What can you say about the size of the multiplier?

Use the **LU**-decomposition to compute the *inverse* of $\mathbf{H}_2$.

Verify your answer.

What is the condition number of $\mathbf{H}_2$?

SOLUTION : Here

$$\mathbf{H}_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} ,$$

for which

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} , \quad \mathbf{U} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{12} \end{pmatrix} , \quad \text{and} \quad \mathbf{H}_2^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix} .$$

The sole multiplier that arises in the **LU**-decomposition has value $\frac{1}{2}$, and

$$\text{cond}(\mathbf{H}_2) = \| \mathbf{H}_2 \|_\infty \ \| \mathbf{H}_2^{-1} \|_\infty = \frac{3}{2} \cdot 18 = 27 .$$

PROBLEM 10 :

Compute (pencil and paper) the **LU**-decomposition of the $n \times n$ *Hilbert matrix* $\mathbf{H}_n$ whose entries in the ith row and jth column are

$$h_{i,j} \;=\; 1/(i+j-1), \quad i,j = 1, \cdots, n \ ,$$

for the case $n \;=\; 3$.

Use the **LU**-decomposition to solve

$$\mathbf{H}_3 \, \mathbf{x} \;=\; \mathbf{f} \ ,$$

for $\mathbf{x}$, when $\mathbf{f} \;=\; (0,0,1)^T$, *i.e.*, first solve

$$\mathbf{L} \, \mathbf{g} \;=\; \mathbf{f} \ ,$$

followed by solving $\mathbf{U} \, \mathbf{x} \;=\; \mathbf{g}$.

SOLUTION : Here

$$\mathbf{H}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} ,$$

for which

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} , \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{pmatrix} .$$

The solution of $\mathbf{H}_3\,\mathbf{x} = \mathbf{f}$, when $\mathbf{f} = (0,\ 0,\ 1)^T$, is $\mathbf{x} = (30, 180, 180)^T$.

The matrix $\mathbf{H}_n$ becomes very *ill-conditioned* as its dimension n increases.

PROBLEM 11 :

Suppose that *solving a general linear system* of equations of dimension n requires 0.1 second on a given computer when $n = 10^2$.

Based on the number of operations (multiplications and divisions only) estimate how much time it will take to *multiply* two general square matrices of dimension 10^3 on this computer.

SOLUTION :

The *leading term* of the number of "operations" (multiplications and divisions) for solving a system of linear equations is $n^3/3$.

Solving a linear system of dimension $n = 100$ takes 0.1 second.

Thus each operation takes $0.1/(10^6/3) = 3 \cdot 10^{-7}$ seconds.

Multiplying two n by n matrices takes n^3 operations (multiplications).

Multiplying two matrices of dimension $n = 1000$ takes 10^9 operations.

The estimated time to multiply two matrices of dimension $n = 1000$ is

$$10^9 \cdot 3 \cdot 10^{-7} = 300 \text{ seconds.}$$

PROBLEM 12 :

Let $\mathbf{T}_n$ be the specific n by n *tridiagonal matrix* $\mathbf{T}_n = \text{diag}[2, 5, 2]$.

What upper bound on $\text{cond}(\mathbf{T}_n)$ is obtained,

when making use of the Banach Lemma?

SOLUTION :

$$\operatorname{cond}(\mathbf{T}_n) \ \leq \ 9 \ .$$

PROBLEM 13 :

Let $\mathbf{T}_n$ be *any general* n *by* n *tridiagonal matrix.*

What is the total of the number of multiplications and divisions needed to determine the **LU**-decomposition of $\mathbf{T}_n$?

(Note that there is no right-hand side vector $\mathbf{f}$.)

SOLUTION :

The total number of "operations" (multiplications and divisions) is $2n-2$.

PROBLEM 14 :

If Newton's method is used to compute *the cube root of* 3,

with initial guess $x^{(0)} = 1$, then what will be the value of $x^{(1)}$?

SOLUTION : $x^{(1)} = \frac{5}{3}$.

PROBLEM 15 :

Use the methods below to compute $\sqrt{2}$, *i.e.*, solve the equation

$$x^2 - 2 \ = \ 0 \ ,$$

for its positive root.

In particular, determine the number of iterations k needed so that the *residual* $| \, (x^{(k)})^2 - 2 \, |$ is less than 10^{-5}.

- *Newton's method* with $x^{(0)} = 1$.

- The *Chord method* *with* $x^{(0)} = 1$.

SOLUTION :

For Newton's Method:

k	x	residual
1	0.15000000D+01	0.25000000D+00
2	0.14166667D+01	0.69444444D-02
3	0.14142157D+01	0.60073049D-05

For the Chord Method:

k	x	residual
1	0.15000000D+01	0.25000000D+00
2	0.13750000D+01	-0.10937500D+00
3	0.14296875D+01	0.44006348D-01
4	0.14076843D+01	-0.18424838D-01
5	0.14168967D+01	0.75963863D-02
.	...	...
.	...	...
12	0.14142079D+01	-0.15913179D-04
13	0.14142159D+01	0.65914283D-05

PROBLEM 16 :

Given $x^{(0)}$, say, $x^{(0)} = 2.0$, compute the sequence

$$x^{(1)}\ ,\ \ x^{(2)}\ ,\ \ x^{(3)}\ ,\ \ \cdots\ ,\ \ x^{(N)}\ ,$$

up to a large value of N, *e.g.*, $N = 10$, using the recurrence relation

$$x^{(k+1)} = f(x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \ \cdots,$$

where

$$f(x) \ = \ \frac{x^2 + 5}{2x}\ .$$

Describe in a few words the observed behavior of the sequence.

In particular, does the sequence approach a limiting value?

If yes, then do you recognize what this limiting value is?

Does the limiting value depend on $x^{(0)}$?

SOLUTION : The sequence quickly *converges* :

```
k       x
1   0.30000000D+01
2   0.23333333D+01
3   0.22380952D+01
4   0.22360689D+01
5   0.22360680D+01
6   0.22360680D+01
7   0.22360680D+01
```

The limiting value $x^* \approx 2.2360680$ is a *fixed point*, *i.e.*, $f(x^*) = x^*$.

Thus x^* is a solution of

$$x = \frac{x^2+5}{2x} ,$$

from which we find that $x^2 = 5$.

Indeed the limiting value 2.2360680 is the positive square root of 5,

and we recognize the iteration as Newton's method for the root of 5.

PROBLEM 17 :

Consider the recurrence relation

$$x^{(k+1)} = cx^{(k)}(1 - x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots\ .$$

Prove that if $c \in [0,4]$ and $x^{(0)} \in [0,1]$ then $x^{(k)} \in [0,1]$ for all k.

SOLUTION :

$$x^{(k+1)} = f(x^{(k)}) \ , \qquad \text{where} \qquad f(x) \ = \ cx(1-x) \ .$$

The curve $y = f(x)$ is a *parabola* , with $f(x) = 0$ at $x = 0$ and $x = 1$.

Furthermore, $f'(x) = 0$ at $x = \frac{1}{2}$, where $f(x)$ reaches its maximum value, namely, $f(\frac{1}{2}) \ = \ \frac{c}{4}$.

(Draw the graph of $y = f(x)$ in the x-y-plane !)

Thus if c has value in the interval $[0, 4]$, and if $x^{(k)}$ is in the interval $[0, 1]$, then $x^{(k+1)} = f(x^{(k)})$ also is in the interval $[0, 1]$.

Since $x^{(0)}$ is taken in the interval $[0, 1]$ it follows by induction that $x^{(k)}$ lies in the interval $[0, 1]$, for all k.

PROBLEM 18 :

Consider the recurrence relation

$$x^{(k+1)} = cx^{(k)}(1 - x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots\ .$$

For each of these values of c :

$$c \ = \ 0.5\ ,\ 1.5\ ,\ 3.5\ ,$$

- analytically determine all fixed points.
- analytically determine whether or not they are attracting.

SOLUTION :

The fixed points are the solutions of $x = f(x)$

i.e., of

$$x = c\,x\,(1-x)\ ,$$

namely,

$$x^* = 0\ , \quad \text{and} \quad x^* = 1 - \frac{1}{c}\ .$$

A fixed point x^* is "*attracting*" (or "*stable*") if $|\,f'(x^*)\,| < 1$.

Here

$$f(x) = c\,x\,(1-x), \quad \text{and} \quad f'(x) = c\,(1-2x)\ ,$$

The conclusions are given in the Table on the following page.

c	x^*	$\mid f'(x^*) \mid$	stable?	x^*	$\mid f'(x^*) \mid$	stable?
0.5	0	0.5	Yes			
1.5	0	1.5	No	$\frac{1}{3}$	0.5	Yes
3.5	0	3.5	No	$\frac{5}{7}$	1.5	No

PROBLEM 19 :

For given $x^{(0)}$, say, $x^{(0)} = 3.10$, compute

$$x^{(1)},\ x^{(2)},\ x^{(3)},\ \cdots\ ,\ x^{(N)}\ ,$$

up to a suitably large value of N, using the recurrence relation

$$x^{(k+1)} \ = \ \tan(x^{(k)})\ , \qquad k = 0,\ 1,\ 2,\ \cdots\ .$$

Does this sequence have a limit?

Can you explain the observed behavior?

Do the same for $x^{(0)} = 6.2828$.

SOLUTION :

A reasonably accurate graph will explain the seemingly *complex behaviour* of this fixed point iteration:

- Draw a graph that includes several of the "*branches*" of the tangent function $f(x) = \tan(x)$.

- Note the *zero intercepts* of the branches, namely at $x = k\,\pi$, for $k = \cdots -3, -2, -1, 0, 1, 2, \cdots$.

- Also note the *vertical asymptotes* at $x = (2k+1)\,\pi/2$.

- Include the line $\ell(x) = x$ in the graph, and observe its *intersects* with each of the infinitely many branches of $f(x) = \tan(x)$.

- Thus there are *infinitely many fixed points*, including $x = 0$.

- Note that $f'(x) = 1$ at the fixed point $x = 0$, *i.e.*, the derivative test is *inconclusive* for $x = 0$.

- However, graphical interpretation of the iteration shows that the fixed point $x = 0$ is repelling, albeit *weakly repelling*.

- Note that all of the other infinitely many fixed points are *repelling*.

- Thus *all* infinitely many fixed points are *repelling*, which explains the complex behavior.

- Finally note that if $x^{(k)}$ is close to $x = 0$ for some k, then it can take many more iterations before they leave the neighborhood of $x = 0$.

PROBLEM 20 :

Determine all fixed points of this iteration:

$$x^{(k+1)} = f(x^{(k)}) ,$$

where

$$f(x) = e^x .$$

SOLUTION :

A graph of $f(x) = e^x$ and $\ell(x) = x$, shows that there is *no fixed point.*

PROBLEM 21 :

Determine all fixed points of this iteration:

$$x^{(k+1)} = f(x^{(k)}) ,$$

where

$$f(x) = e^{-x} .$$

SOLUTION :

A graph of $f(x) = e^{-x}$ and $\ell(x) = x$ shows that there is one fixed point, namely,

$$x^* \approx 0.56714329 ,$$

and that x^* is attracting, with

$$| f'(x^*) | < 1 ,$$

but

$$f'(x^*) \neq 0 .$$

Thus for sufficiently close $x^{(0)}$ there will be *linear convergence* .

PROBLEM 22 :

Determine all fixed points of this iteration:

$$x^{(k+1)} \;=\; f(x^{(k)}) \;,$$

where

$$f(x) \;=\; \frac{1+x^2}{1+x} \;.$$

For each fixed point determine whether it is attracting.

SOLUTION : A fixed point of

$$x^{(k+1)} = f(x^{(k)}) , \quad \text{with} \quad f(x) = \frac{1+x^2}{1+x} ,$$

satisfies

$$x = \frac{1+x^2}{1+x} ,$$

which can be rewritten as

$$x + x^2 = 1 + x^2 ,$$

from which it follows that $x^* = 1$ is the only fixed point.

The derivative is seen to be

$$f'(x) = \frac{x^2 + 2x - 1}{(x+1)^2} , \quad \text{with} \quad |\, f'(x^*) \,| = \frac{1}{2} < 1 .$$

Thus $x^* = 1$ is *attracting* for sufficiently close $x^{(0)}$.

PROBLEM 23 :

Consider the Chord method for solving the equation $x^2 - 2 = 0$:

$$x^{(k+1)} \ = \ x^{(k)} - \frac{(x^{(k)})^2 - 2}{2x_c} \ ,$$

Here x_c is a constant, $x_c \ \neq \ 0$.

(Normally one chooses x_c close to the square root of 2 , and $x^{(0)} \ = \ x_c$.)

What are the fixed points of this iteration ?

For each fixed point determine all values of x_c for which the fixed point is attracting.

SOLUTION :

Here

$$x^{(k+1)} = f(x^{(k)}) \ , \quad \text{with} \quad f(x) = x - \frac{x^2 - 2}{2x_c} \ ,$$

and

$$f'(x) = 1 - \frac{x}{x_c} \ .$$

We see that for $x^* = +\sqrt{2}$:

$$|\, f'(x^*) \,| < 1 \quad \text{if} \quad x_c > \frac{1}{2}\sqrt{2} \ ,$$

while for $x^* = -\sqrt{2}$:

$$|\, f'(x^*) \,| < 1 \quad \text{if} \quad x_c < -\frac{1}{2}\sqrt{2} \ .$$

PROBLEM 24 :

Consider the function $g(x) = x^2 - 3$.

- Write down Newton's method for finding a zero of $g(x)$.
- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method.
- Will Newton's method converge for all initial points ?
- To which zero does it converge (as dependent on the initial guess) ?

SOLUTION :

For $g(x) = x^2 - 3$:

- Write down Newton's method for finding a zero of $g(x)$:

$$x^{(k+1)} = f(x^{(k)}) \quad \text{where} \quad f(x) = x - \frac{x^2 - 3}{2x} = \frac{x^2 + 3}{2x} \; .$$

- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method :

  TO BE DONE !

- Will Newton's method converge for all initial points ?

  All initial points, except $x = 0$ which give a division by zero.

- To which zero does it converge (as dependent on the initial guess) ?

  To $+\sqrt{2}$ for positive $x^{(0)}$, and to $-\sqrt{2}$ for negative $x^{(0)}$.

PROBLEM 25 :

Now consider the function $g(x) = x^3 - 3$:

- Write down Newton's method for finding a zero of $g(x)$.

- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method.

- Will Newton's method converge for all initial points ?

  (Hint: *This is a little more difficult to answer here* !)

SOLUTION :

For $g(x) = x^3 - 3$ (which has only one root) :

- Write down Newton's method for finding a zero of $g(x)$:

$$x^{(k+1)} = f(x^{(k)}) ,$$

where

$$f(x) = x - \frac{x^3 - 3}{3x^2} = \frac{2x^3 + 3}{3x^2} .$$

- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method :

  TO BE DONE !

- From the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" we see that :

  - Newton's method converges for all positive $x^{(0)}$.

  - The initial point $x^{(0)} = 0$ gives a *division by zero* .

  - Newton's method converges for *almost all* negative $x^{(0)}$.

  - A countably infinite number of negative $x^{(0)}$ result in *division by zero* .

    (These reach $x = 0$ after a finite number of iterations.)

PROBLEM 26 :

Consider Newton's method for solving the *system of equations*

$$\begin{aligned} x_1 \ - \ e^{-x_2} \ &= \ 0 \ , \\ 2\ e^{-x_1} - x_2 \ &= \ 0 \ . \end{aligned}$$

Use $x_1^{(0)} = 0$ and $x_2^{(0)} = 0$ as initial guesses.

Determine $x_1^{(1)}$ and $x_2^{(1)}$.

SOLUTION :

For the first iteration of Newton's Method we have :

$$\begin{pmatrix} 1 & e^{-x_2^{(0)}} \\ -2e^{-x_1^{(0)}} & -1 \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix} \;=\; -\begin{pmatrix} x_1^{(0)} \;-\; e^{-x_2^{(0)}} \\ 2e^{-x_1^{(0)}} \;-\; x_2^{(0)} \end{pmatrix} \;,$$

which, with $x_1^{(0)} = 0$ and $x_1^{(0)} = 0$, becomes

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ -2 \end{pmatrix} \;,$$

from which

$$\begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;,$$

so that

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} \;=\; \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} \;+\; \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;.$$

PROBLEM 27 : Lagrange Interpolation

Suppose that $f(x) = e^x$, and that the interpolation points $\{x_k\}_{k=0}^n$ are distinct, but otherwise arbitrary in the interval $[-1, 1]$.

How big must n be, so that

$$|\, p_n(x) - e^x \,| \leq 10^{-6} \text{ for all } x \in [-1, 1] \text{ ?}$$

SOLUTION : The question is for what value of n is $\frac{2^{n+1}}{(n+1)!} \cdot e < 10^{-6}$?

Here is a simple fortran code that determines the answer:

```
 IMPLICIT NONE
 DOUBLE PRECISION e, r, bound
 INTEGER k, n

 n = 14
 e = DEXP(1.0D0)

! The ratio when n=0:
 r = 2.d0

 DO k = 1,n
   r = 2*r/(k+1)
   WRITE(6,101) k, r
 ENDDO

 bound = e * r
 WRITE(6,102)n, bound

 101 FORMAT(1X," k =",I3,3X,"ratio =",D12.5)
 102 FORMAT(/" n =",I3,3X,"bound =",D12.5)
 STOP
 END
```

SOLUTION (continued) : The output is:

```
k =  1    ratio = 0.20000D+01
k =  2    ratio = 0.13333D+01
k =  3    ratio = 0.66667D+00
k =  4    ratio = 0.26667D+00
k =  5    ratio = 0.88889D-01
k =  6    ratio = 0.25397D-01
k =  7    ratio = 0.63492D-02
k =  8    ratio = 0.14109D-02
k =  9    ratio = 0.28219D-03
k = 10    ratio = 0.51307D-04
k = 11    ratio = 0.85511D-05
k = 12    ratio = 0.13156D-05
k = 13    ratio = 0.18794D-06
k = 14    ratio = 0.25058D-07

n = 14    bound = 0.68115D-07
```

PROBLEM 28 :

Consider the unique interpolating polynomial p_4 of degree 4 or less that interpolates the function $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| f - p_4 \|_\infty ,$$

for the case where:

The points $\{x_k\}_{k=0}^4$ are distinct in $[-1, 1]$, but they are otherwise arbitrary.

(Use a tight bound on $\| \prod_{k=0}^4 (x - x_k) \|_\infty$ in your derivation.)

SOLUTION :

$$\| f - p_4 \|_\infty \leq \frac{2^{n+1}}{(n+1)!} .$$

```
n =  1   bound = 2.000000D+00
n =  2   bound = 1.333333D+00
n =  3   bound = 6.666667D-01
n =  4   bound = 2.666667D-01
n =  5   bound = 8.888889D-02
n =  6   bound = 2.539683D-02
n =  7   bound = 6.349206D-03
n =  8   bound = 1.410935D-03
n =  9   bound = 2.821869D-04
n = 10   bound = 5.130672D-05
n = 11   bound = 8.551120D-06
```

PROBLEM 29 :

Again consider the unique interpolating polynomial p_4 of degree 4 or less that interpolates the function $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| f \; - \; p_4 \|_\infty \;\; ,$$

for the case where:

$$x_0 = -1 \;,\;\; x_1 = -0.5 \;,\;\; x_2 = 0 \;,\;\; x_3 = 0.5 \;,\;\; x_4 = 1 \; .$$

(Use a tight bound on $\| \prod_{k=0}^{4}(x - x_k) \|_\infty$ in your derivation.)

SOLUTION : By the Lagrange Interpolation Theorem, with $n = 4$,

$$f(x) - p_4(x) \ = \ \frac{f^{(5)}(\xi)}{5!} \, w_5(x) \ ,$$

for some point $\xi(x) \in [-1, 1]$.

Here the local maxima and minima of

$$w_5(x) \ = \ (x+1) \, (x+\frac{1}{2}) \, x \, (x-\frac{1}{2}) \, (x-1)$$

can be found analytically, namely at

```
x =-0.822216     w(x) = 0.113482
x =-0.271956     w(x) =-0.044334
x = 0.271956     w(x) = 0.044334
x = 0.822216     w(x) =-0.113482
```

Thus

$$|\, f(x) - p_4(x) \,| \ \leq \ \frac{0.113482}{120} \ = \ 9.45683 \ 10^{-4} \ .$$

PROBLEM 30 :

Once more consider the interpolating polynomial p_4 of degree 4 or less that interpolates $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| \, f \ - \ p_4 \, \|_\infty \ ,$$

when the points $\{x_k\}_{k=0}^4$ are the roots of $T_5(x)$ (Chebyshev points).

(Use a tight bound on $\| \prod_{k=0}^4 (x - x_k) \, \|_\infty$ in your derivation.)

SOLUTION :

$$| \, f(x) - p_4(x) \, | \;\; \leq \;\; \frac{2^{-n}}{(n+1)!} \;=\; \frac{2^{-4}}{5!} \;=\; \frac{1}{1920} \;\approx\; 5.20833 \; \cdot \; 10^{-4} \; .$$

PROBLEM 31 : Taylor's Theorem

If $f \in C^{n+1}[a,b]$ and $x \in [a,b]$, then $f(x) = p_n(x) + R_n(x)$, where

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \ (x - x_0)^k$$

is the *Taylor polynomial*, and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \ (x - x_0)^{n+1}, \qquad \text{for some } \xi(x) \in [a,b]$$

is the *Taylor Remainder* (or "error term").

What are $p_n(x)$ and $R_n(x)$, when $f(x) = e^x$, and $x_0 = 0$?

SOLUTION :

$$p_n(x) \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \cdots \;+\; \frac{x^n}{n!} \;,$$

and

$$R_n(x) \;=\; \frac{x^{n+1}}{(n+1)!}\; e^{\xi(x)} \;,$$

where

$$\xi(x) \text{ lies between } 0 \text{ and } x \;.$$

(Here x can be taken to be positive or negative.)

PROBLEM 32 :

For the *Taylor polynomial* $p_n(x)$ of degree n for $f(x) = e^x$ about $x_0 \ = \ 0$,

what is the smallest value of $\ n$, so that

$$| \ e^x - p_n(x) \ | \quad < \ 10^{-3} \ ,$$

everywhere in the interval $[-1, 1]$?

SOLUTION :

Here

$$| \, e^x \; - \; p_n(x) \, | \quad = \quad | \, R_n(x) \, | \quad = \quad | \, \frac{x^{n+1}}{(n+1)!} \, | \; e^{\xi(x)} \quad \leq \quad \frac{e}{(n+1)!} \; ,$$

which is seen to be less than 10^{-3} when $n \geq 6$:

```
n = 1:   b = 1.359141D+00
n = 2:   b = 4.530470D-01
n = 3:   b = 1.132617D-01
n = 4:   b = 2.265235D-02
n = 5:   b = 3.775391D-03
n = 6:   b = 5.393416D-04
```

PROBLEM 33 :

Derive the Taylor polynomial p_n of degree n for $f(x) = \sin(x)$

about the point $x_0 = 0$, for the case $n = 5$.

Draw a reasonably accurate graph of $p_5(x)$, together with $f(x)$.

Use the Taylor Theorem to derive an upper bound on

$$\| f - p_5 \|_\infty \ ,$$

for the interval $[-1, 1]$.

SOLUTION :

Here

$$p_5(x) \;=\; x \;-\; \frac{x^3}{6} \;+\; \frac{x^5}{120} \;,$$

and

$$|\; p_5(x) \;-\; \sin(x) \;| \;=\; |\; \frac{x^7}{7!}\; \cos(\xi(x)) \;| \;\leq\; \frac{1}{5040} \;<\; 2 \cdot 10^{-4} \;.$$

Why is using the error term

$$|\; \frac{x^7}{7!}\; \cos(\xi(x)) \;\;|$$

correct here ?