ASSIGNMENT PROBLEMS

for

ELEMENTARY NUMERICAL METHODS

PROBLEM 1 :

Let $\mathbf{x} \in \mathbb{R}^3$ be given by $\mathbf{x} = (3, 4, -5)^T$.

What is the value of $\| \mathbf{x} \|_2$?

PROBLEM 2 :

Use the Banach Lemma (with the matrix infinity norm) to prove that the n by n matrix $\mathbf{T}_n$ given below is invertible for all positive integers n :

$$\mathbf{T}_n \;=\; \mathrm{diag}[1,1,5,1,1] \;\equiv\; \begin{pmatrix} 5 & 1 & 1 & & & & \\ 1 & 5 & 1 & 1 & & & \\ 1 & 1 & 5 & 1 & 1 & & \\ & 1 & 1 & 5 & 1 & 1 & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & 1 & 5 & 1 \\ & & & & 1 & 1 & 5 \end{pmatrix} .$$

Hint: First rewrite $\mathbf{T}_n$ as $\mathbf{T}_n = c\,\tilde{\mathbf{T}}_n$, where c is a constant, chosen so that the Banach Lemma can be applied to $\tilde{\mathbf{T}}_n$.

Also use the Banach Lemma to derive an upper bound on the infinity norm of the inverse matrix $\mathbf{T}_n^{-1}$, and on the condition number of $\mathbf{T}_n$.

PROBLEM 3 :

Use the Banach Lemma (with the matrix infinity norm) to prove that the n by n matrix $\mathbf{S}_n$ given below is invertible for all positive integers n :

$$\mathbf{S}_n \; = \; \mathrm{diag}[h_i, 2(h_i + h_{i+1}), h_{i+1}] \; \equiv$$

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ & h_2 & 2(h_2+h_3) & h_3 & & \\ & & h_3 & 2(h_3+h_4) & h_4 & \\ & & \cdot & \cdot & \cdot & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & h_{n-1} & 2(h_{n-1}+h_n) \end{pmatrix},$$

where $h_i > 0$, for all i. (This matrix arises in cubic spline interpolation.)

Hint: First rewrite $\mathbf{S}_n$ as $\mathbf{S}_n = \mathbf{D}_n \, \tilde{\mathbf{S}}_n$, where $\mathbf{D}_n$ is a diagonal matrix, chosen so that the Banach Lemma can be applied to $\tilde{\mathbf{S}}_n$.

PROBLEM 4 :

Use Gauss elimination to compute (pencil and paper) the **LU**-decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} .$$

After having computed **L** and **U**, use them to solve for $\mathbf{x}$ in

$$\mathbf{A}\mathbf{x} = \mathbf{f} \ ,$$

where $\mathbf{f} = (2, 2, 1)^T$.

Check your answer !

PROBLEM 5 :

Give an example of a *singular* 2×2 matrix $\mathbf{A}$:

that *has* an **LU**-decomposition.

Show **L** and **U**.

PROBLEM 6 :

Give an example of a *nonsingular* 2×2 matrix $\mathbf{A}$:

that does *not have* an **LU**-decomposition.

Explain why not.

PROBLEM 7 :

A square matrix $\mathbf{A}$ is said to be *ill-conditioned* if the *condition number*

$$\text{cond}(\mathbf{A}) \equiv \| \mathbf{A} \| \ \| \mathbf{A}^{-1} \| \ ,$$

is large.

Give an example of a *nonsingular* 2×2 matrix $\mathbf{A}$ that is *ill-conditioned*, with

$$\text{cond}(\mathbf{A}) \geq 10^6 \ ,$$

but where the multiplier that arises in the **LU**-decomposition is not big in absolute value.

PROBLEM 8 :

Give an example of a 2×2 matrix $\mathbf{A}$ that is *not ill-conditioned*, with

$$\text{cond}(\mathbf{A}) \ \le \ 10 \ ,$$

but where the multiplier that arises in the **LU** -decomposition is of very large magnitude, *i.e.*, is big in absolute value.

PROBLEM 9 :

Compute (pencil and paper) the **LU**-decomposition of the $n \times n$ *Hilbert matrix* $\mathbf{H}_n$ whose entries in the ith row and jth column are

$$h_{i,j} \;=\; 1/(i+j-1), \quad i,j = 1, \cdots, n \;,$$

for the case $n \;=\; 2$.

What can you say about the size of the multiplier?

Use the **LU**-decomposition to compute the *inverse* of $\mathbf{H}_2$.

Verify your answer.

What is the condition number of $\mathbf{H}_2$?

PROBLEM 10 :

Compute (pencil and paper) the **LU**-decomposition of the $n \times n$ *Hilbert matrix* $\mathbf{H}_n$ whose entries in the ith row and jth column are

$$h_{i,j} \ = \ 1/(i+j-1), \quad i,j = 1, \cdots, n \ ,$$

for the case $n \ = \ 3$.

Use the **LU**-decomposition to solve

$$\mathbf{H}_3 \, \mathbf{x} \ = \ \mathbf{f} \ ,$$

for $\mathbf{x}$, when $\mathbf{f} \ = \ (0,0,1)^T$, *i.e.*, first solve

$$\mathbf{L} \, \mathbf{g} \ = \ \mathbf{f} \ ,$$

followed by solving $\mathbf{U} \, \mathbf{x} \ = \ \mathbf{g}$.

PROBLEM 11 :

Suppose that *solving a general linear system* of equations of dimension n requires 0.1 second on a given computer when $n = 10^2$.

Based on the number of operations (multiplications and divisions only) estimate how much time it will take to *multiply* two general square matrices of dimension 10^3 on this computer.

PROBLEM 12 :

Let $\mathbf{T}_n$ be the specific n by n *tridiagonal matrix* $\mathbf{T}_n \ = \ \text{diag}[2, 5, 2]$.

What is the upper bound on $\text{cond}(\mathbf{T}_n)$ is obtained,

when making use of the Banach Lemma?

PROBLEM 13 :

Let $\mathbf{T}_n$ be *any general* n *by* n *tridiagonal matrix.*

What is the total of the number of multiplications and divisions needed to determine the **LU**-decomposition of $\mathbf{T}_n$?

(Note that there is no right-hand side vector $\mathbf{f}$.)

PROBLEM 14 :

If Newton's method is used to compute *the cube root of* 3,

with initial guess $x^{(0)} = 1$,

then what will be the value of $x^{(1)}$?

PROBLEM 15 :

Use the methods below to compute $\sqrt{2}$, *i.e.*, solve the equation

$$x^2 - 2 = 0 ,$$

for its positive root.

In particular, determine the number of iterations k needed so that the *residual* $| (x^{(k)})^2 - 2 |$ is less than 10^{-5}.

- *Newton's method* with $x^{(0)} = 1$.

- The *Chord method* *with* $x^{(0)} = 1$.

PROBLEM 16 :

Given $x^{(0)}$, say, $x^{(0)} = 2.0$, compute the sequence

$$x^{(1)}\ ,\ \ x^{(2)}\ ,\ \ x^{(3)}\ ,\ \ \cdots\ ,\ \ x^{(N)}\ ,$$

up to a large value of N, *e.g.*, $N = 10$, using the recurrence relation

$$x^{(k+1)} = f(x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots,$$

where

$$f(x) \ = \ \frac{x^2 + 5}{2x}\ .$$

Describe in a few words the observed behavior of the sequence.

In particular, does the sequence approach a limiting value?

If yes, then do you recognize what this limiting value is?

Does the limiting value depend on $x^{(0)}$?

PROBLEM 17 :

Consider the recurrence relation

$$x^{(k+1)} = cx^{(k)}(1 - x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots \ .$$

Prove that if $c \in [0, 4]$ and $x^{(0)} \in [0, 1]$ then $x^{(k)} \in [0, 1]$ for all k.

PROBLEM 18 :

Consider the recurrence relation

$$x^{(k+1)} = cx^{(k)}(1 - x^{(k)}), \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots\ .$$

For each of these values of c :

$$c \ = \ 0.5\ ,\ 1.5\ ,\ 3.5\ ,$$

- analytically determine all fixed points.
- analytically determine whether or not they are attracting.

PROBLEM 19 :

For given $x^{(0)}$, say, $x^{(0)} = 3.10$, compute

$$x^{(1)},\ x^{(2)},\ x^{(3)},\ \cdots\ ,\ x^{(N)}\ ,$$

up to a suitably large value of N, using the recurrence relation

$$x^{(k+1)} \ = \ \tan(x^{(k)})\ , \qquad k = 0,\ 1,\ 2,\ \cdots\ .$$

Does this sequence have a limit?

Can you explain the observed behavior?

Do the same for $x^{(0)} = 6.2828$.

PROBLEM 20 :

Determine all fixed points of this iteration:

$$x^{(k+1)} = f(x^{(k)}) \ ,$$

where

$$f(x) = e^x \ .$$

PROBLEM 21 :

Determine all fixed points of this iteration:

$$x^{(k+1)} = f(x^{(k)}) \ ,$$

where

$$f(x) = e^{-x} \ .$$

PROBLEM 22 :

Determine all fixed points of this iteration:

$$x^{(k+1)} \;=\; f(x^{(k)}) \;,$$

where

$$f(x) \;=\; \frac{1+x^2}{1+x} \;.$$

For each fixed point determine whether it is attracting.

PROBLEM 23 :

Consider the Chord method for solving the equation $x^2 - 2 = 0$:

$$x^{(k+1)} \ = \ x^{(k)} - \frac{(x^{(k)})^2 - 2}{2x_c} \ ,$$

Here x_c is a constant, $x_c \ \neq \ 0$.

(Normally one chooses x_c close to the square root of 2 , and $x^{(0)} \ = \ x_c$.)

What are the fixed points of this iteration ?

For each fixed point determine all values of x_c for which the fixed point is attracting.

PROBLEM 24 :

Consider the function $g(x) = x^2 - 3$.

- Write down Newton's method for finding a zero of $g(x)$.

- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method.

- Will Newton's method converge for all initial points ?

- To which zero does it converge (as dependent on the initial guess) ?

PROBLEM 25 :

Now consider the function $g(x) = x^3 - 3$:

- Write down Newton's method for finding a zero of $g(x)$.

- Draw the "$x^{(k+1)}$ versus $x^{(k)}$ diagram" for Newton's method.

- Will Newton's method converge for all initial points ?

  (Hint: *This is a little more difficult to answer here* !)

PROBLEM 26 :

Consider Newton's method for solving the *system of equations*

$$x_1 - e^{-x_2} = 0 ,$$

$$2\, e^{-x_1} - x_2 = 0 .$$

Use $x_1^{(0)} = 0$ and $x_2^{(0)} = 0$ as initial guesses.

Determine $x_1^{(1)}$ and $x_2^{(1)}$.

PROBLEM 27 : Lagrange Interpolation

If $f \in C^{n+1}[a,b]$ and $x \in [a,b]$, then

$$f(x) \ = \ p_n(x) \ + \ R_n(x) \ ,$$

where

$$p_n(x) \ = \ \textstyle\sum_{k=0}^{n} \ f(x_k) \ l_k(x) \ .$$

The polynomials $l_k(x)$ are the *Lagrange interpolating coefficients* , and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k), \qquad \text{for some} \ \ \xi(x) \in [a,b] \ ,$$

is the *Lagrange remainder* (error term).

The *interpolation points* x_i are distinct, but otherwise arbitrary.

If $f(x) \ = \ e^x$, then how big must n be, so that

$$| \ p_n(x) - e^x \ | \ \ \leq \ 10^{-6} \ \ \text{for all} \ \ x \in [-1,1] \ ?$$

PROBLEM 28 :

Consider the unique interpolating polynomial p_4 of degree 4 or less that interpolates the function $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| f - p_4 \|_\infty ,$$

for the case where:

The points $\{x_k\}_{k=0}^4$ are distinct in $[-1, 1]$, but they are otherwise arbitrary.

(Use a tight bound on $\| \prod_{k=0}^4 (x - x_k) \|_\infty$ in your derivation.)

PROBLEM 29 :

Again consider the unique interpolating polynomial p_4 of degree 4 or less that interpolates the function $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| f - p_4 \|_\infty ,$$

for the case where:

$$x_0 = -1 , \; x_1 = -0.5 , \; x_2 = 0 , \; x_3 = 0.5 , \; x_4 = 1 .$$

(Use a tight bound on $\| \prod_{k=0}^4 (x - x_k) \|_\infty$ in your derivation.)

PROBLEM 30 :

Once more consider the interpolating polynomial p_4 of degree 4 or less that interpolates the function $f(x) = \sin(x)$ at five distinct points $\{x_0, x_1, x_2, x_3, x_4\}$ in the interval $[-1, 1]$.

Use the Lagrange Interpolation Theorem to derive an upper bound on

$$\| f - p_4 \|_\infty \ ,$$

for the case where:

The points $\{x_k\}_{k=0}^4$ are the roots of $T_5(x)$ (Chebyshev points).

(Use a tight bound on $\| \prod_{k=0}^4 (x - x_k) \|_\infty$ in your derivation.)

PROBLEM 31 :

Taylor's Theorem :

If $f \in C^{n+1}[a,b]$ and $x \in [a,b]$, then $f(x) = p_n(x) + R_n(x)$, where

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \, (x - x_0)^k$$

is the *Taylor polynomial*, and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \, (x - x_0)^{n+1}, \qquad \text{for some } \xi(x) \in [a,b]$$

is the *Taylor Remainder* (= error term).

What are $p_n(x)$ and $R_n(x)$, when $f(x) = e^x$, and $x_0 = 0$?

PROBLEM 32 :

For the *Taylor polynomial* $p_n(x)$ of degree n for $f(x) = e^x$ about $x_0 = 0$, what is the smallest value of n, so that

$$| e^x - p_n(x) | < 10^{-3} ,$$

everywhere in the interval $[-1, 1]$?

PROBLEM 33 :

Derive the Taylor polynomial p_n of degree n for $f(x) = \sin(x)$

about the point $x_0 = 0$, for the case $n = 5$.

Draw a reasonably accurate graph of $p_5(x)$, together with $f(x)$.

Use the Taylor Theorem to derive an upper bound on

$$\| f - p_5 \|_\infty \ ,$$

for the interval $[-1, 1]$.

PROBLEM 34 :

Let $x_0 = 0$, $x_1 = h$, and $x_2 = 2h$ be uniformly spaced points.

Derive a *numerical differentiation formula* for

$$f'(x_0) \text{ , in terms of } f(x_0),\ f(x_1), \text{ and } f(x_2).$$

Taylor expand to determine a formula for the error in this approximation.

PROBLEM 35 :

Derive a *five-point approximation formula* for the *third derivative* of a sufficiently smooth function $f(x)$.

Use the *reference interval* $[-2h, 2h]$, with

$$x_0 = -2h \ , \quad x_1 = -h \ , \quad x_2 = 0 \ , \quad x_3 = h \ , \quad x_4 = 2h \ .$$

The formula should approximate $f'''(x_2)$ in terms of the values of $f(x)$ at the points x_0, x_1, x_2, x_3, and x_4.

Use Taylor expansions to determine the *leading error term* of the formula.

If $f(x) = e^x$ then how small should h be so the error is less than 10^{-6} ?

PROBLEM 36 :

Determine the *order of accuracy* of the following 3-point, asymmetric, numerical differentiation formula:

$$f''(0) \cong \frac{f(2h) - 2f(h) + f(0)}{h^2} .$$

PROBLEM 37 :

For the *Lagrange polynomial* $p_n(x)$ of degree n or less, that interpolates

$$f(x) = e^x \quad \text{at} \quad n+1 \quad \textit{Chebyshev points} \text{ in the interval } [-1, 1] \ ,$$

what is the smallest value of n so that

$$| \ e^x - p_n(x) \ | < 10^{-3} \ ,$$

everywhere in $[-1, 1]$?

PROBLEM 38 :

Suppose we approximate $f(x) = \sin(x)$ in the interval $[0, 2\pi]$

by local polynomial interpolation at 3 *Chebyshev points* ,

using *local polynomials* of degree 2 or less.

What is the *smallest number of intervals of equal size* needed

so that the maximum interpolation error is less than 10^{-3} ?

PROBLEM 39 :

How many *function evaluations* are needed by

the composite Simpson integration formula ,

to integrate a function $f(x)$ over an interval $[a, b]$,

based on local integration in N subintervals of size $h = (b - a)/N$.

PROBLEM 40 :

Determine the *order of accuracy* of the *composite Simpson formula* to integrate a sufficiently differentiable function $f(x)$ over an interval $[a, b]$, based on local integration in N subintervals of size $h = (b - a)/N$?

PROBLEM 41 :

The *local Trapezoidal Rule* for the reference interval $[-h/2, h/2]$ is

$$\int_{-h/2}^{h/2} f(x)\ dx \ \cong \ \frac{h}{2}\Big[f(-h/2) + f(h/2)\Big] \ .$$

Use Taylor expansions to derive *the local error formula* .

Let $h = (b-a)/N$ and $x_k = a + k\ h$, for $k = 0, 1, 2, 3, \ \cdots \ , N$.
Then the *composite Trapezoidal Rule* is given by

$$\int_a^b f(x)dx \ \cong \ \frac{h}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)\Big].$$

Based on the local error, derive *an upper bound on the global error* .

PROBLEM 42 :

Use the *Gram-Schmidt procedure* to construct an orthogonal basis

of the 3-dimensional linear space

$$\mathcal{E}_3 \equiv \text{Span}\{ 1 , x^2 , x^4 \}$$

for the case of the interval $[-1, 1]$.

Determine the *best approximation* $p^*(x) \in \mathcal{E}_3$ to the function $f(x) = x^6$.

What is the value of $\| f(x) - p^*(x) \|_2$?

NOTE: "*best approximation*" means "best approximation in the $\| \cdot \|_2$ ".

PROBLEM 43 :

Show that the functions

$$e_0(x) \equiv 1, \;\; e_1(x) = \sin(x), \;\; e_2(x) = \cos(x), \; e_3(x) = \sin(2x), \;\; e_4(x) = \cos(2x)$$

are mutually *orthogonal* with respect to the inner product

$$< f \;,\; g > \;\; = \;\; \int_0^{2\pi} \; f(x) \; g(x) \; dx \; .$$

Also show how one can determine the coefficients c_k , $k = 0, 1, 2, 3, 4,$

of the *trigonometric polynomial*

$$p(x) = c_0 + c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) + c_4 \cos(2x) \; ,$$

that, for a given function $f(x)$, minimizes the integral

$$\int_0^{2\pi} \; (\; p(x) - f(x) \;)^2 \; dx \; .$$