ELEC 261

Lecture 1, July 4, 2006

Definition of a Complex number

$z = x + iy$ with x and y real and

$i = \sqrt{-1}$ or $i^2 = -1$

is a <u>complex</u> number.

Two complex numbers $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$

are equal if $x_1 = x_2$

and $y_1 = y_2$

<u>Arithmetic operations</u>

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

<u>Commutative Law</u>: $z_1 + z_2 = z_2 + z_1$

$z_1 z_2 = z_2 z_1$

Associative Law: $z_1 + (z_2 + z_3) + (z_1 + z_2) + z_3$

$z_1(z_2 z_3) = (z_1 z_2) z_3$

Distributive Law:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

Example: Add $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$

$$(2 + 4i) + (-3 + 8i) = -1 + 12i$$

Example: Multiply $z_1 = 2 - 3i$ and $z_2 = 4 + 5i$

$$z_1 z_2 = (2 - 3i)(4 + 5i) = 8 - 12i + 10i + 15$$
$$= 23 - 2i$$

Conjugate: $z = x + iy$

$\bar{z} = x - iy$ is called the Conjugate of z.

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$$

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Note that:

$$z + \bar{z} = x + iy + x - iy = 2x$$

So

$$x = Re[z] = \frac{z + \bar{z}}{2}$$

Similarly $y = Im(z) = \dfrac{z - \bar{z}}{2i}$

<u>Example</u>: Divide $z_1 = 2-3i$ by $z_2 = 4+6i$

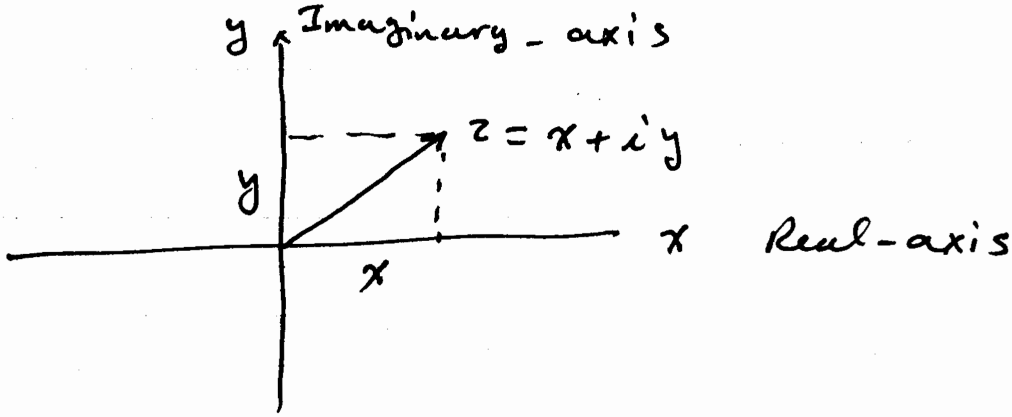
$$\frac{z_1}{z_2} = \frac{2-3i}{4+6i} = \frac{(2-3i)(4-6i)}{16+36}$$

$$= \frac{8-12i-12i+18i^2}{52}$$

$$= \frac{-10-24i}{52}$$

$$\boxed{= -\frac{5}{26} - \frac{6}{13}i}$$

<u>Geometric Interpretation</u>



<u>Length</u> of z or <u>Modulus</u> of z or <u>Absolute value</u> of z is:

$$|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

Example: Find the length of (Modulus of)

$$z = 4 - 3i$$

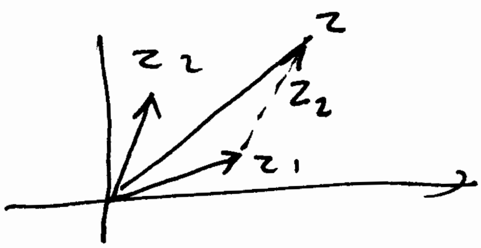
$$|z| = \sqrt{16 + 9} = 5$$

$$z\bar{z} = (4-3i)(4+3i) = 16 - 12i + 12i - 9i^2$$
$$= 16 + 9 = 25$$

$$|z| = \sqrt{z\bar{z}} = 5$$

Triangle inequality

Let $z = z_1 + z_2$



Then $|z_1 + z_2| \leq |z_1| + |z_2|$

in general

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

$$|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |z_2|$$

or $|z_1 + z_2| \geq |z_1| - |z_2|$

Example:

Find: $\frac{1}{z}$ if $z = 2+3i$

$$\frac{1}{z} = \frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{4+\cancel{6i}-\cancel{6i}+9}$$

$$\frac{1}{z} = \frac{2-3i}{13} = \boxed{\frac{2}{3} - \frac{3}{13}i}$$

Example: Calculate

$$\frac{(1+i)(1-2i)}{(2+i)(4-3i)}$$

$$\frac{(1+i)(1-2i)}{(2+i)(4-3i)} = \frac{1+i-2i+2}{8+4i-6i+3} = \frac{3-i}{11-2i} = \frac{(3-i)(11+2i)}{11^2+2^2}$$

$$= \frac{33-11i+6i+2}{125} = \frac{35-5i}{125} = \frac{7-i}{25} = \boxed{\frac{7}{25} - \frac{1}{25}i}$$

Example:

Show that for all points z on the circle $x^2+y^2=4$, we have

$$|z+6+8i| \leq 12$$

Solution

$\cancel{|z+6+8i| = |x+iy+6+8i|} = |(x+6)+i(y+8)|$

$= \sqrt{(x+6)^2+(y+8)^2} \cancel{\leq} \ldots = \sqrt{64+406}$

$= \sqrt{\cancel{3}} \leq \ldots$

$$|z+6+8i| \leq |z|+|6+8i| = 2+\sqrt{36+64} = 12$$

Example: Take $z_1 = 10+8i$

and $z_2 = 11-6i$

which one is closer to origin

$$\begin{array}{r} 121 \\ 36 \\ \hline 157 \end{array}$$

$$|z_1| = \sqrt{100+64} = \sqrt{164}$$

$$|z_2| = \sqrt{11^2+36} = \sqrt{157}$$

$|z_2| < |z_1|$ so z_2 is closer to zero.

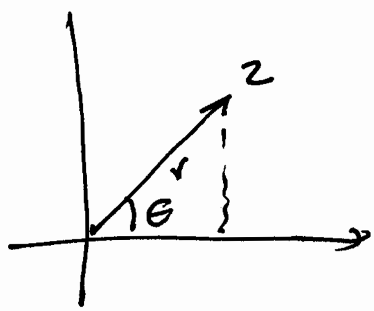
Polar form of Complex numbers

Let $z = x+iy$

then let

$$r = \sqrt{x^2+y^2}$$

and $\theta = \operatorname{atan}\left(\frac{y}{x}\right)$



then

$$z = r\cos\theta + r\sin\theta\, i$$

$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Example:

Express $z = 1 - \sqrt{3}\, i$ in polar form

$$r = |z| = \sqrt{1 + (\sqrt{3})^2} = 2$$

$$\tan(\theta) = \frac{-\sqrt{3}}{1} = \frac{5\pi}{3} = \arg(z)$$

So

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$$

Multiplication and Division:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

So

$$|z_1 z_2| = |z_1||z_2|$$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

we could also derive ~~this~~ directly

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big]$$

$$= r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

Division

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

So

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Example: Find the polar form of $z = i$

$|z| = 1$ $\qquad \tan(\theta) = \frac{y}{x} = \frac{1}{0} = \infty$

$$\theta = \frac{\pi}{2} \Rightarrow z = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Example: Let $z_1 = i$ and $z_2 = 1-\sqrt{3}\,i$

Find polar form of $z_1 z_2$ and z_1/z_2.

$$|z_1 z_2| = r_1 r_2 = 2\times 1 = 2$$

$$\begin{aligned}\operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)\\ &= \frac{\pi}{2} + \frac{5\pi}{3} = \frac{3\pi+10\pi}{6} = \frac{13\pi}{6} = 2\pi + \frac{\pi}{6} = \frac{\pi}{6}\end{aligned}$$

So

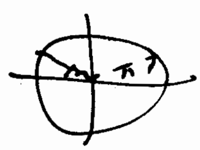
$$\begin{aligned} z_1 z_2 = 2e^{i\pi/6} &= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\ &= 2\left(\frac{\sqrt{3}}{2} + i\,\tfrac{1}{2}\right) = \sqrt{3} + i\end{aligned}$$

Similarly (directly):

$$z_1 z_2 = i(1-\sqrt{3}\,i) = \sqrt{3} + i$$

$$\begin{aligned}\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) &= \frac{\pi}{2} - \frac{5\pi}{3}\\ &= \frac{3\pi - 10\pi}{6} = -\frac{7\pi}{6}\\ &= \frac{5\pi}{6}\end{aligned}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{1}{2}$$



So

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\\ &= \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{\sqrt{3}}{4} + \frac{i}{4}\end{aligned}$$

Directly

$$\frac{z_1}{z_2} = \frac{i}{1-\sqrt{3}i} = \frac{i(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = \frac{i-\sqrt{3}}{4}$$

$$= -\frac{\sqrt{3}}{4} + \frac{i}{4}$$

Powers of z:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

We saw that

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

if $z_1 = z_2 = z$ then

$$z^2 = r^2 [\cos(2\theta) + i\sin(2\theta)]$$

in general

$$z^n = r^n [\cos(n\theta) + i\sin(n\theta)]$$

<u>Example</u>: Find z^3 if $z = 1 - \sqrt{3}\, i$

$$z = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$$

$$\text{or} = 2\left[\cos\frac{5\pi}{3} + i\sin\left(\frac{5\pi}{3}\right)\right]$$

$$z^3 = 8\left[\cos 5\pi + i\sin 5\pi\right]$$

$$= 8\left[\cos\pi + i\sin\pi\right] = \boxed{-8}$$

n-th root of a complex number

$$z = r\left[\cos(\theta) + i\sin(\theta)\right] = w^n$$

where $w = \rho\left[\cos(\alpha) + i\sin(\alpha)\right]$

Then

$$w^n = \rho^n\left[\cos(n\alpha) + i\sin(n\alpha)\right] =$$

$$= r\left[\cos(\theta) + i\sin(\theta)\right)$$

So

$$\rho = r^{1/n}$$

and $n\alpha = \theta + 2m\pi \Rightarrow \alpha = \frac{\theta + 2m\pi}{n}$

So, any non-zero complex number z has n nth-roots as

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$

Example: Find the cube roots of $z = i$

$r = 1$ and $\theta = \arg z = \frac{\pi}{2}$

$$w_k = (1)^{1/3}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right]$$

$k = 0, 1, 2$

So

$$w_0 = \cos\frac{\pi}{6} + i\sin\pi/6 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_1 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_2 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

Example: Find fourth-roots of $z = 1+i$

$$r = \sqrt{1+1} = \sqrt{2} \qquad \theta = \frac{\pi}{4}$$

$$w_k = (2)^{1/4}\left[\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{4}\right)\right]$$

$$k = 0, 1, 2, 3$$

$$w_0 = \sqrt{2}^{1/4}\left[\cos \pi/16 + i \sin \pi/16\right] = 1.1664 + 0.2320\, i$$

$$w_1 = \sqrt{2}^{1/4}\left[\cos \frac{9\pi}{16} + i \sin 9\pi/16\right] = -0.2320 + 1.1664\, i$$

$$w_2 = \sqrt{2}^{1/4}\left[\cos \frac{17\pi}{16} + i \sin \frac{17\pi}{16}\right] = -1.1664 - 0.2320\, i$$

$$w_3 = \sqrt{2}^{1/4}\left[\cos \frac{25\pi}{16} + i \sin \frac{25\pi}{16}\right] = 0.2320 - 1.1664\, i$$

Example: Find the 6th-roots of $z = 1$

$$z = 1 \Rightarrow |z| = r = 1 \quad , \theta = 0$$

$$w_k = 1^{1/6}\left[\cos\left(\frac{2\pi k}{6}\right) + i \sin\left(\frac{2\pi k}{6}\right)\right] \qquad k = 0, 1, 2, \cdots 5$$

$$w_0 = 1$$

$$w_1 = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + i\, \sqrt{3}/2$$

$$w_2 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + i\, \sqrt{3}/2$$

$$w_3 = \cos \pi + i \sin \pi = -1$$

$$w_4 = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -1/2 - i\, \sqrt{3}/2$$

$$w_5 = \cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}$$

$$w_5 = 1/2 - i\,\frac{\sqrt{3}}{2}$$