Lecture 10, August 3, 2006

Residues

Given a Laurent expansion of $f(z)$ as,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

in $0 < |z-z_0| < R$, we call a_{-1} the residue of $f(z)$ at z_0 and show it as

$$a_{-1} = Res(f(z), z_0)$$

Example: $f(z) = \dfrac{1}{(z-1)^2(z-3)}$

we can write this as

$$f(z) = -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) + \cdots$$

in $0 < |z-1| < 2$.

So, the residue of $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ at $z=1$ is $-\frac{1}{4}$ or $Res(f(z), 1) = -\frac{1}{4}$.

Example:

$$f(z) = e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$$

in $|z| > 0$

So, $a_{-1} = 3$ and $Res(f(z), 0) = 3$

Example: $f(z) = \dfrac{1}{z(z-1)} = \dfrac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$

in $0 < |z-1| < 1$.

So $\text{Res}(f(z), 1) = 1$

Observation: if we multiply both sides by $(z-1)$ we get

$$(z-1) f(z) = \frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

now, if we let $z=1$ we get

$$(z-1) f(z)\Big|_{z=1} = \frac{1}{z}\Big|_{z=1} = 1 - 0 + 0^2 - \cdots$$

So

$$\text{Res}(f(z), 1) = (z-1) f(z)\Big|_{z=1}$$

in general if $z = z_0$ is a simple pole of $f(z)$ then,

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

<u>Example</u>: Find residue of $f(z) = \dfrac{1}{z(z-1)}$ at $z=0$

$z f(z) = \dfrac{1}{z-1} \to -1$ as $z \to 0$

$$\text{Res}(f(z), 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{1}{z-1} = -1$$

Recall that

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \cdots \quad \text{for } 0 < |z| < 1$$

In general:

Theorem: If f has a pole of order n at $z = z_0$, then

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z)$$

Example: Find the residue of

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

at $z=1$. Note that $z=1$ is a pole of order 2 so, $n=2$

$$Res(f(z), 1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} (z-1)^2 f(z)$$

$$= \lim_{z \to 1} \frac{d}{dz} \frac{1}{z-3} = \lim_{z \to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}$$

An alternative residue formula for simple poles:

Assume that $f(z) = \frac{g(z)}{h(z)}$ where $g(z)$ and $f(z)$ are analytic at $z = z_0$

Then,

$$Res(f(z), z_0) = \lim_{z \to 0} (z-z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}}$$

we have used the fact that $h(z_0)=0$

Note that $\lim_{z\to z_0} \frac{h(z)-h(z_0)}{z-z_0} = h'(z_0)$

So

$$Re\,(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

Example: Compute residues of

$$f(z) = \frac{1}{z^4+1}$$

at its poles.

$f(z)=\infty$ when $z^4+1=0 \Rightarrow z^4=-1=e^{i(2k\pi+\pi)}$

$$z = e^{i\frac{2k\pi+\pi}{4}} = e^{i(\frac{k\pi}{2}+\frac{\pi}{4})} \qquad k=0,1,2,3$$

So,

$$z_1 = e^{i\frac{\pi}{4}}, \quad z_2 = e^{i\frac{3\pi}{4}}, \quad z_3 = e^{i\frac{5\pi}{4}}, \quad z_4 = e^{i\frac{7\pi}{4}}$$

That is, $f(z)$ has four simple poles at these points.

$$Res(f(z), z_1) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{-1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$Res(f(z), z_2) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-i\frac{9\pi}{4}} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$Res(f(z), z_3) = \frac{1}{4z_3^3} = \frac{1}{4}e^{-i\frac{15\pi}{4}} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$Res(f(z), z_4) = \frac{1}{4z_4^3} = \frac{1}{4}e^{-i\frac{21\pi}{4}} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

Why did we introduce residues and spent time learning how to compute residues? The next theorem answers this question:

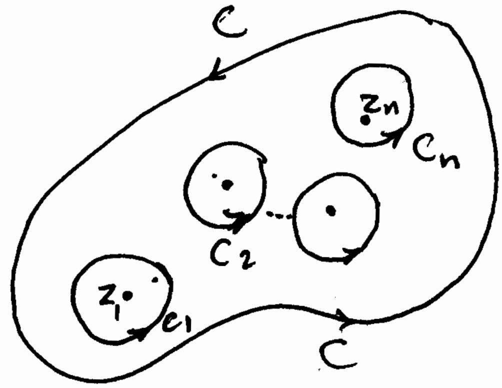
Cauchy's Residue Theorem:

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If function f is analytic on and inside C, except at a finite number of points $z_1, z_2, \cdots z_n$ inside C, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

Proof: Let $C_1, C_2, \cdots C_n$ be circles centered at $z_1, z_2, \cdots z_n$, respectively. Assume that the radius of each of these circles, r_k, is small enough so that $C_1, C_2, \cdots C_n$ are mutually disjoint and are entirely inside C. Then

$\oint_{C_k} f(z)\,dz = 2\pi i a_{-1} = 2\pi i \,\mathrm{Res}(f(z), z_k)$.



Then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

Example: Evaluate

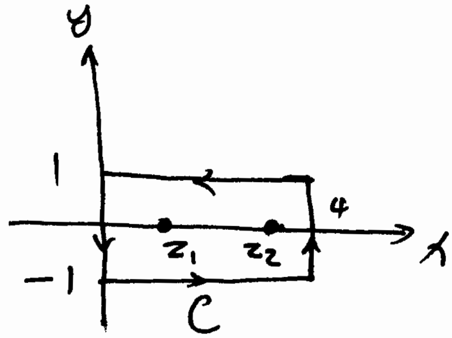
$$\oint_C \frac{dz}{(z-1)^2(z-3)}$$

for

a) C: $x=0,\ x=4,\ y=-1,\ y=1$

b) C: $|z|=2$

a)



both $z=1$ and $z=3$ are inside C, so:

$$\oint \frac{dz}{(z-1)^2(z-3)} = [\mathrm{Res}(f(z),1) + \mathrm{Res}(f(z),3)]\,2\pi i$$

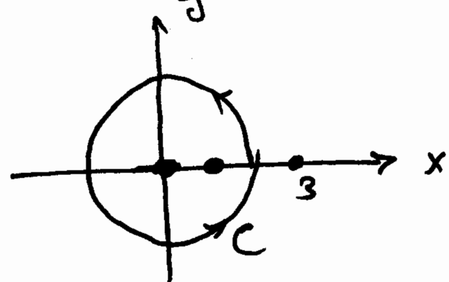
$\mathrm{Res}(f(z),1) = -\frac{1}{4}$

$\mathrm{Res}(f(z),3) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$

So,

$$\oint_C \frac{dz}{(z-1)^2(z-3)} = 2\pi i \left(\frac{-1}{4} + \frac{1}{4}\right) = 0$$

b)



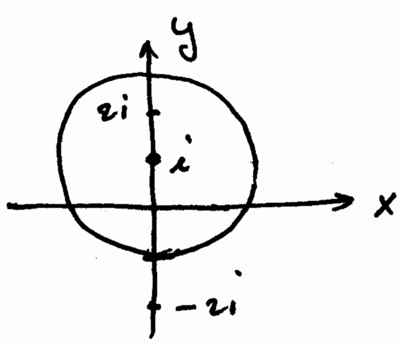
only $z=1$ is inside C. So,

$$\oint \frac{dz}{(z-1)^2(z-3)} = 2\pi i \,\mathrm{Res}(f(z),1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2} i$$

<u>Example</u>: Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$

where C is $|z-i| = 2$

$z^2+4=0 \Rightarrow z_1 = 2i, \quad z_2 = -2i$



only $z_1 = 2i$ is inside C. So,

$$\oint_C \frac{2z+6}{z^2+4} = 2\pi i \,\mathrm{Res}(f(z), 2i)$$

$$\operatorname{Res}(f(z), 2i) = \lim_{z \to 2i} (z-2i) \frac{2z+6}{(z-2i)(z+2i)}$$

$$= \frac{6+4i}{4i} = \frac{3+2i}{2i}$$

Therefore,

$$\oint_C \frac{2z+6}{z^2+4} dz = 2\pi i \left(\frac{3+2i}{2i}\right) = \pi(3+2i)$$

<u>Example:</u> Evaluate $\oint_C \frac{e^z}{z^4+5z^3} dz$

where C is the circle $|z| = 2$

$$z^4 + 5z^3 = 0 \Rightarrow z = 0 \text{ and } z = -5$$

But only $z=0$ is inside C. So,

$$\oint_C \frac{e^z}{z^4+5z^3} dz = 2\pi i \operatorname{Res}(f(z), 0)$$

$$= \frac{2\pi i}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} z^3 \frac{e^z}{z^3(z+5)}$$

$$= \pi i \lim_{z \to 0} \frac{(z^2+8z+17)}{(z+5)^3} e^z = \boxed{\frac{17\pi}{125} i}$$

Example: Evaluate $\oint_C \tan z \, dz$ over $|z| = 2$.

Write $\tan z = \frac{\sin z}{\cos z}$. So, $\tan z$'s poles are

where $\cos z = 0 \Rightarrow z = \frac{\pi}{2} + k\pi \quad k = 0, \pm 1, \pm 2, \ldots$

Out of all poles, only $z_1 = \frac{\pi}{2}$ and $z_2 = -\frac{\pi}{2}$ are inside C.

So,

$$\oint_C \tan z\,dz = 2\pi i\left[\mathrm{Res}\left(f(z),\frac{\pi}{2}\right)+\mathrm{Res}\left(f(z),-\frac{\pi}{2}\right)\right]$$

$$\mathrm{Res}\left(f(z),\frac{\pi}{2}\right)=\frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})}=-1$$

$$\mathrm{Res}\left(f(z),-\frac{\pi}{2}\right)=\frac{\sin(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2})}=-1$$

Therefore,

$$\oint_C \tan z\,dz = 2\pi i(-1-1)=-4\pi i$$

Example: Evaluate $\oint_C f(z)\,dz$

where $f(z)=e^{3/z}$ and C is $|z|=1$

$f(z)$ has an essential singularity. So, we cannot use the residue formulas for finite order singularity.

Therefore, we use direct expansion:

$$e^{3/z}=1+\frac{3}{z}+\frac{3^2}{2!z^2}+\frac{3^3}{3!z^3}+\cdots \qquad |z|>0$$

We see that $\mathrm{Res}(e^{3/z},0)=3$

So,

$$\oint_C e^{3/z}\,dz = 2\pi i\,\mathrm{Res}(e^{3/z},0)=2\pi i(3)=\boxed{6\pi i}$$