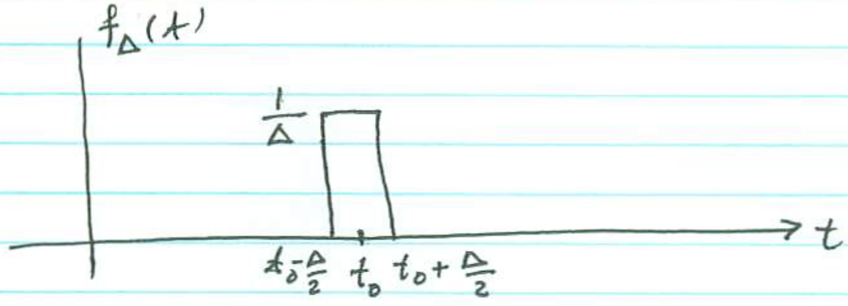
Lecture 11, August 8, 2006

A strange "function":

Consider the following function



That is:

$$f_\Delta(t) = \begin{cases} 0 & t < t_0 - \frac{\Delta}{2} \\ \frac{1}{\Delta} & t_0 - \frac{\Delta}{2} \leq t \leq t_0 + \frac{\Delta}{2} \\ 0 & t > t_0 + \frac{\Delta}{2} \end{cases}$$

Take integral of $f_\Delta(t)$

$$\int_{-\infty}^{\infty} f_\Delta(t) = \int_{t_0-\frac{\Delta}{2}}^{t_0+\frac{\Delta}{2}} \frac{1}{\Delta}\,dt = 1$$

note that the integral is equal to 1 irrespective of the value of Δ.

Now, let $\Delta \to 0$. Then, $f_\Delta(t)$ would be zero everywhere except at $t = t_0$ where it is equal to ∞. But still its integral is equal to one. This limiting function is called Dirac's delta function.

$$\lim_{\Delta \to 0} f_\Delta(t) = \delta(t - t_0) = \begin{cases} \infty & t \neq t_0 \\ 0 & t = 0 \end{cases}$$

But

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

Now, let $f(t)$ be an arbitrary function.

$$f(t)\, f_\Delta(t) = \begin{cases} \frac{1}{\Delta} f(t) & t_0 - \frac{\Delta}{2} \leq t \leq t_0 + \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\, f_\Delta(t)\, dt = \int_{t_0 - \frac{\Delta}{2}}^{t_0 + \frac{\Delta}{2}} \frac{1}{\Delta} f(t)\, dt$$

When Δ is very small $\quad f(t) \cong f(t_0),\ t_0 - \frac{\Delta}{2} \leq t \leq t_0 + \frac{\Delta}{2}$

So,

$$\int_{-\infty}^{\infty} f(t)\, f_\Delta(t)\, dt = \frac{1}{\Delta} \int_{t_0 - \frac{\Delta}{2}}^{t_0 + \frac{\Delta}{2}} f(t_0)\, dt_0 = f(t_0)$$

So, as $\Delta \to 0$

$$\boxed{\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)}$$

This is called the sifting property of the delta function.

Note that we do not need to have integral extended from $-\infty$ to ∞.

Integral between any two values a and b is equal to $f(t_0)$ as long as $a < t_0 < b$

$$\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0) \qquad \text{if } a<t_0<b$$

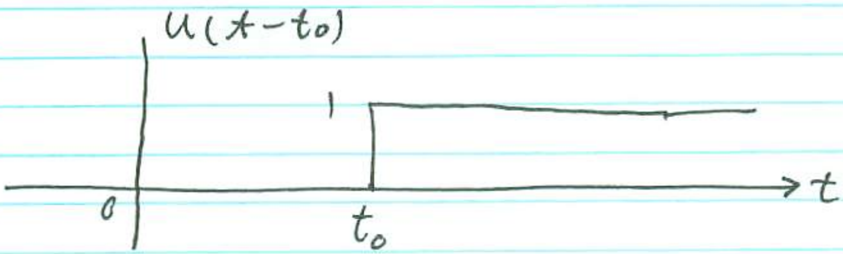
but,
$$\int_a^b f(t)\,\delta(t-t_0)\,dt = 0$$

if either $a \leq b < t_0$ or $t_0 < a \leq b$.

$\delta(t-t_0)$ is also called an impulse function.

Note that

$$\int_{-\infty}^{t} \delta(t'-t_0)\,dt' = \begin{cases} 0 & t<t_0 \\ 1 & t>t_0 \end{cases} = u(t-t_0)$$



This is called the unit step function.

So

$$\int_{-\infty}^{t} \delta(t'-t_0)\,dt' = u(t-t_0)$$

and

$$\frac{d}{dt} u(t-t_0) = \delta(t-t_0)$$

When $t_0 = 0$, we have

$$\int_{-\infty}^{\infty} \delta(t')\,dt' = u(t)$$

and

$$\frac{d}{dt} u(t) = \delta(t).$$

<u>Example</u>: Evaluate the following integral

$$\int_{-\infty}^{\infty} (t^2+3t+5)\,\delta(t-2)\,dt$$

$$\int_{-\infty}^{\infty} (t^2+3t+5)\,\delta(t-2) = (t^2+3t+5)\Big|_{t=2}$$

$$= 2^2+3\times 2+5 = 15$$

The Convolution / Linear-Time-invariant Systems

Recal that

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

replace t by τ to get

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(\tau-t_0)\,d\tau = f(t_0)$$

Substituting t for t_0, we get,

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$

where, we have used the fact that

$$\delta(t-\tau) = \delta(\tau-t).$$

Note that, the equation

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

indicates that $f(t)$ Can be represented as an infinite sum of delta functions each weighted by the value of $f(t)$ at the center of the delta function.

Now assume that we have a system that inputs a function (say a waveform), $x(t)$ and outputs another waveform $y(t)$.

$$x(t) \longrightarrow \boxed{\;T\;} \longrightarrow y(t)$$

That is

$$T[x(t)] = y(t)$$

A system is called <u>linear</u> if its output for sum of two inputs is equal to sum of the two outputs to those two inputs, i.e.,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

A system is called <u>time-invariant</u> if its output to a shifted version of a waveform is a shifted version of its output to that input. That is, if

$$T[x(t)] = y(t)$$

then $$T[x(t-t_0)] = y(t-t_0)$$

We saw that any function can be written as a sum of delta functions. So, for a <u>linear time-invariant</u> system, if we know the output for a delta (impulse) function, then we can find the output for any arbitrary function. That is if we have

$$T[\delta(t)] = h(t)$$

or

$$\delta(t) \longrightarrow \boxed{\;T\;} \longrightarrow h(t)$$

$h(t)$ is called the <u>impulse response</u> of the system

then

$$y(t) = T[x(t)] = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

This is called the convolution of $x(t)$ and $h(t)$.

If $x(t)$ is causal, i.e., $x(t) = 0,\ t < 0$

then,

$$y(t) = \int_0^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

also if $h(t)$ is causal, i.e., $h(t) = 0$

for $t < 0$ then

$$h(t-\tau) = 0 \text{ for } t - \tau < 0$$
$$\text{or } \tau > t.$$

So: $$\boxed{y(t) = \int_0^{t} x(\tau)\, h(t-\tau)\, d\tau} = x(t) * h(t)$$

The <u>Laplace</u> <u>Transforms</u>

Laplace transform is used to turn a Convolution into a multiplication of two functions. It is also used to solve differential Equations into algebraic equations and making the solution of ~~d~~ differential equations easy.

<u>Definition</u>: Let $f(t)$ be a function defined for $t \geq 0$. Then

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

is called the <u>Laplace</u> <u>transform</u> of $f(t)$, provided that the integral exists.

<u>Example</u>: Find the Laplace transform of $\delta(t)$

$$\mathcal{L}[\delta(t)] = \int_0^\infty e^{-st}\,\delta(t)\,dt = e^{-st}\Big|_{t=0} = 1$$

<u>Example</u>: Find the Laplace transform of $u(t)$.

Remember

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

Example: Find the Laplace transform of

$f(t) = t$

Note: here we mean $f(t) = t$ for $t > 0$ and $f(t) = 0$ for $t < 0$. So, it would be more formal if we write:

$$f(t) = t\, u(t).$$

Solution

$$\mathcal{L}[t] = \int_0^\infty t e^{-st}\, dt = \cancel{\frac{-te^{-st}}{s}}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s} \times \frac{-e^{-st}}{s}\Big|_0^\infty = \frac{1}{s^2}$$

Example: Evaluate $\mathcal{L}\{e^{-3t}\}$

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-3t} e^{-st}\, dt = \int_0^\infty e^{-(3+s)t}\, dt$$

$$= \frac{-1}{s+3} e^{-(s+3)t}\Big|_0^\infty = \boxed{\frac{1}{s+3}}$$

for $s > -3$.

Example: Find the Laplace transform of

$f(t) = \sin \omega t$

$$\mathcal{L}[\sin \omega t] = \int_0^\infty \sin\omega t\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\, e^{-st}\, dt$$

$$= \frac{1}{2i}\int_0^\infty e^{(i\omega - s)t}\, dt - \frac{1}{2i}\int_0^\infty e^{-(i\omega + s)t}\, dt$$

$$= \frac{1}{2i}\frac{1}{i\omega - s} e^{(i\omega - s)t}\Big|_0^\infty + \frac{1}{2i}\frac{1}{i\omega + s} e^{-(i\omega+s)t}\Big|_0^\infty$$

if $s > 0$

$$= \frac{1}{2i}\frac{1}{i\omega - s}(-1) + \frac{1}{2i}\frac{1}{i\omega + s}(-1)$$

$$= -\frac{1}{2i}\left[\frac{1}{i\omega - s} + \frac{1}{i\omega + s}\right]$$

$$= -\frac{1}{2i}\left[\frac{i\omega + s + i\omega - s}{-\omega^2 - s^2}\right] = \frac{\omega}{s^2 + \omega^2}$$

Example: Find the Laplace transform of

$f(t) = \cos \omega t$

$$\mathcal{L}[\cos \omega t] = \int_0^\infty \cos \omega t \, e^{-st} \, dt = \int_0^\infty \frac{e^{i\omega t} + e^{-i\omega t}}{2} e^{-st} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-i\omega)t} \, dt + \frac{1}{2} \int_0^\infty e^{-(s+i\omega)t} \, dt$$

$$= \frac{1}{2} \frac{-1}{s-i\omega} e^{-(s-i\omega)t} \Big|_0^\infty + \frac{1}{2} \frac{-1}{(s+i\omega)} e^{-(s+i\omega)t} \Big|_0^\infty$$

$$= \frac{-1}{2} \frac{1}{s-i\omega} (0-1) - \frac{1}{2} \frac{1}{s+i\omega} (0-1)$$

$$= \frac{1}{2} \left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right] = \frac{1}{2} \frac{2s}{s^2+\omega^2}$$

$$= \boxed{\frac{s}{s^2+\omega^2}}$$

Example: Find the Laplace transform of

$f(t) = 1 + 5t$

$$\mathcal{L}[1+5t] = \mathcal{L}[1] + 5\mathcal{L}[t]$$

$$= \boxed{\frac{1}{s} + \frac{5}{s^2}}$$

In the above example, we used the <u>linearity</u> of Laplace transform. That is:

$$\mathcal{L}[a f(t) + b g(t)] = \int_0^\infty [a f(t) + b g(t)] e^{-st} dt$$

$$= a\int_0^\infty f(t) e^{-st} dt + b\int_0^\infty g(t) e^{-st} dt$$

$$= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

Laplace transform of some basic functions:

1) $\mathcal{L}[c] = \frac{c}{s}$, in particular $\mathcal{L}[1] = \frac{1}{s}$

2) $\mathcal{L}[t] = \frac{1}{s^2}$

3) $\mathcal{L}[t^2] = \frac{2}{s^3}$

4) in general, $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

5) $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$

6) $\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}$

7) $\mathcal{L}[\sinh(\omega t)] = \frac{\omega}{s^2-\omega^2}$

8) $\mathcal{L}[\cosh(\omega t)] = \frac{s}{s^2-\omega^2}$

Definition: A function $f(t)$ is said to be of exponential order c if there exist constants c, $M>0$ and $T>0$ such that

$$|f(t)| \leq M e^{ct} \quad \text{for all } t > T.$$

Example $f(t) = t$

$$|t| \leq e^{t} \quad \text{for all } t > 0$$

So $f(t) = t$ is of exponential order $c=1$.

Example: $f(t) = 2\cos t$

$$|2\cos t| \leq 2e^{t} \quad \text{all } t>0$$

So $2\cos t$ is of exponential order $c=1$.

Example: $f(t) = t^{n}$

$$|t^n| \leq M e^{ct} \quad \text{since } \left|\frac{t^n}{e^{ct}}\right| \leq M \text{ for some } M>0$$

This can be shown by using L'Hôpital's rule.

Example: $f(t) = e^{t^2}$ is not of exponential order since it grows faster than any e^{ct}.

Theorem: Sufficient Conditions for existence of Laplace Transform.

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order c for $t > T$, then $\mathcal{L}[f(t)]$ exists for $s > c$.