Lecture 12, August 9, 2006

Inverse Laplace transform:

Definition: If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is said to be the inverse Laplace transform of $F(s)$.

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

Example: Find the inverse Laplace transform of $F(s) = \frac{1}{s}$

We have

$$\mathcal{L}[1] = \frac{1}{s}$$

So,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

Example: Find the inverse Laplace transform of $F(s) = \frac{1}{s^2}$.

Again, using what we saw in the last lecture, i.e.,

$$\mathcal{L}[t] = \frac{1}{s^2}$$

We have

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

We have also the following inverse Laplace transforms:

a) $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{1}{n!}\, t^n$

b) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

c) $\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin \omega t$

d) $\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos \omega t$

e) $\mathcal{L}^{-1}\left[\frac{\omega}{s^2-\omega^2}\right] = \sinh \omega t$

f) $\mathcal{L}^{-1}\left[\frac{s}{s^2-\omega^2}\right] = \cosh \omega t.$

These basic inverse transforms can be used to find more general inverse Laplace transforms.

<u>Example</u>: Find the inverse Laplace transform of $F(s) = \frac{1}{s^2+3s}$

$$F(s) = \frac{1}{s(s+3)} = \frac{1/3}{s} - \frac{1/3}{s+3}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1/3}{s} - \frac{1/3}{s+3}\right]$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t} = \frac{1}{3}(1 - e^{-3t})$$

Example: Find the inverse Laplace transform

of $\quad F(s) = \dfrac{1}{s^3+5s}$

$$F(s) = \frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

$$A = \frac{1}{5}, \quad B = \frac{-1}{5}, \quad C = 0$$

So,

$$F(s) = \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}\right]$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+5}\right]$$

$$= \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}\,t$$

Example: Find $\mathcal{L}^{-1}\left[\frac{-2s+6}{s^2+4}\right]$

$$\mathcal{L}^{-1}\left[\frac{-2s+6}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right]$$

$$= -2\,\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + 3\,\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= -2\cos 2t + 3\sin 2t .$$

Example: Find the inverse Laplace transform of $F(s) = \dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

$$F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

Let $s=2$ to get

$$2^2+6\times 2+9 = B(2-1)(2+4) \Rightarrow B = \frac{25}{6}$$

Let $s=1$ to get

$$1^2+6\times 1+9 = A(1-2)(1+5) \Rightarrow A = -\frac{16}{5}$$

Let $s=-4$ to get $C = +\frac{1}{30}$

So,

$$F(s) = \frac{-16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\mathcal{L}^{-1}[F(s)] = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{3}e^{-4t}$$

Laplace transform of derivative of a function

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sF(s)$$

For second derivative

$$\mathcal{L}[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s[sF(s) - f(0)]$$

$$\boxed{= s^2F(s) - sf(0) - f'(0)}$$

In general

$$\mathcal{L}[f^{(n)}(t)] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Solving linear ordinary differential equations.

Example: Solve

$$\frac{dy}{dt} + 3y = 13\sin 3t, \qquad y(0) = 6$$

$$\mathcal{L}\left[\frac{dy}{dt} + 3y\right] = \mathcal{L}[13\sin 2t]$$

$$\Rightarrow sY(s) - y(0) + 3Y(s) = 13\frac{2}{s^2+4}$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$Y(s) = \frac{\frac{26}{s^2+4} + 6}{s+3} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$Y(s) = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

Example: Solve the following differential equation

$$y'' - 3y' + 2y = e^{-4t} \qquad y(0)=1, \ y'(0)=5$$

$$\mathcal{L}[y'' - 3y' + 2y] = \mathcal{L}[e^{-4t}]$$

$$\mathcal{L}[y''] - 3\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \frac{1}{s+4}$$

$$s^2 Y(s) - s y(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s+4}$$

$$s^2 Y(s) - 3sY(s) + 2Y(s) - s - 5 + 3 = \frac{1}{s+4}$$

$$Y(s) = \frac{\frac{1}{s+4} + s + 2}{s^2 - 3s + 2} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

$$Y(s) = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/3}{s+4}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{3}e^{-4t}$$

Translation theorems:

Translation on the s-axis:

Let $\mathcal{L}[f(t)]$ be $F(s)$

then if we multiply $f(t)$ by e^{at}, we have

$$\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

So, multiplication by an exponential of the form e^{at} in the time domain, results in shift by a in s-domain.

<u>Example</u>: Find the Laplace transform of

$f(t) = e^{5t} t^3$

$$\mathcal{L}[t^3] = \frac{3!}{s^4}$$

$$\mathcal{L}[e^{5t} t^3] = \frac{3!}{(s-5)^4} = \frac{6}{(s-5)^4}$$

Example: Find the inverse Laplace transform of $F(s) = \frac{2s+5}{(s-3)^2}$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

So

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{11}{(s-3)^2}\right]$$

$$= 2\,\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + 11\,\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right]$$

Note that $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

So $\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{+3t}$

similarly,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

So

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right] = t\,e^{3t}$$

So

$$\mathcal{L}^{-1}[F(s)] = 2e^{3t} + 11\,t\,e^{3t}$$

Example: Find

$$\mathcal{L}^{-1}\left[\frac{s/2 + 5/3}{s^2+4s+6}\right]$$

s^2+4s+6 does not have real roots. So, we cannot use partial fractions without having to deal with complex numbers, which is not very easy thing to do. So, we write it as

$$s^2+4s+6 = (s+2)^2 + 2$$

So

$$\mathcal{L}^{-1}\left[\frac{s/2 + 5/3}{s^2+4s+6}\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{2}(s+2) + 2/3}{(s+2)^2 + 2}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+2}\right]$$

Note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right] = \cos\sqrt{2}\,t$$

So $\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2}\right] = e^{-2t}\cos\sqrt{2}\,t$

Also,

$$\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s^2+2)}\right] = \sin\sqrt{2}t$$

So

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+2}\right] = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{2}}{(s+2)^2+2}\right] = \frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{s/2+5/3}{s^2+4s+6}\right] = \frac{1}{2}\cos\sqrt{2}t\, e^{-2t} + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Example: Solve the following differential equation

$$y'' - 6y' + 9y = t^2e^{3t} \qquad y(0)=2,\ y'(0)=6$$

$$\mathcal{L}[y''] - 6\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[t^2e^{3t}]$$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s) - 2s - 6 + 12 = \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{\frac{2}{(s-3)^3} + 2s - 6}{(s-3)^2} = \frac{2}{(s-3)^5} + \frac{2s-6}{(s-3)^2}$$

$$= \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{(s-3)^5} + \frac{2}{s-3}\right]$$

$$y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s-3)^5}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$\mathcal{L}[t^4] = \frac{4!}{s^5}$$

$$\mathcal{L}[e^{3t}t^4] = \frac{4!}{(s-3)^5} = \frac{24}{(s-3)^5}$$

so,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^5}\right] = \frac{1}{24}e^{3t}t^4$$

also

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

so,

$$y(t) = \frac{1}{12}e^{3t}t^4 + 2e^{3t}$$