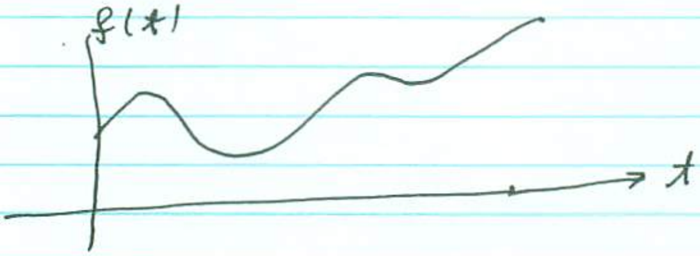
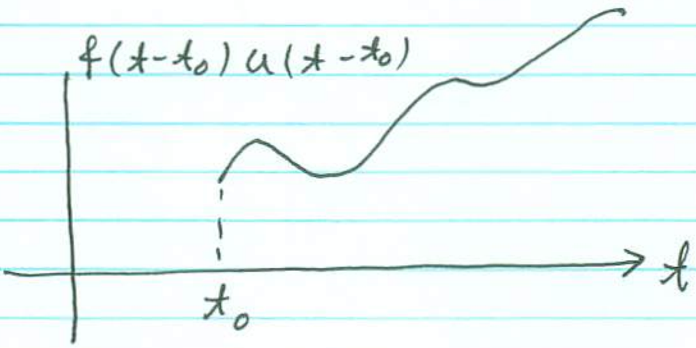
Lecture 13, August 15, 2006

Translation in time-domain

Take the function $f(t)$:

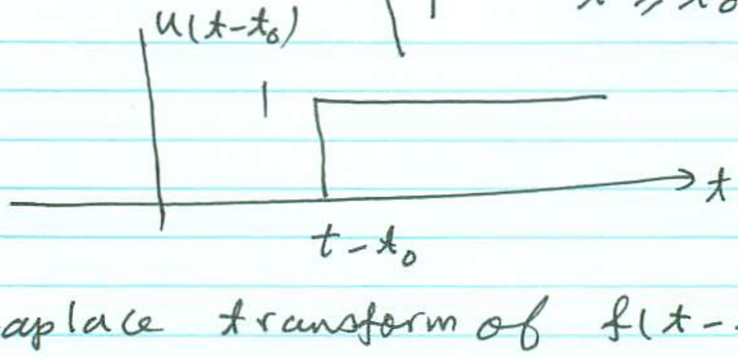


The translated version of it (by t_0) on the t-axis is



where $u(t-t_0)$ is a step function starting at t_0, i.e.,

$$u(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$



The Laplace transform of $f(t-t_0)\,u(t-t_0)$ can be found as follows:

$$\mathcal{L}[f(t-t_0)u(t-t_0)] = \int_0^{\infty} e^{-st} f(t-t_0)u(t-t_0)\,dt$$

$$= \int_{t_0}^{\infty} e^{-st} f(t-t_0)\,dt$$

Let $t - t_0 = \tau \Rightarrow t = \tau + t_0$

Then $dt = d\tau$

Substituting $t = \tau + t_0$ and $dt = d\tau$ in the Laplace transform expression, we get

$$\mathcal{L}[f(t-t_0)] = \int_0^{\infty} e^{-s(\tau+t_0)} f(\tau)\,d\tau$$

$$= e^{-st_0}\int_0^{\infty} e^{-s\tau} f(\tau)\,d\tau = e^{-st_0}F(s)$$

So, we have the following Theorem:

Theorem:

$$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-st_0}F(s)$$

That is, a translation by t_0 in the time domain is equivalent to multiplication by e^{-st_0} in the s-domain. Equivalently, we can write:

$$\mathcal{L}^{-1}[e^{-t_0 s}F(s)] = f(t-t_0)u(t-t_0)$$

Example: Find the inverse Laplace transform of $\frac{s}{s^2+9} e^{-\pi s/2}$

we can write:

$$\frac{s}{s^2+9} e^{-\pi s/2} = F(s) e^{-\frac{\pi s}{2}}$$

where $F(s) = \frac{s}{s^2+9}$

$$\mathcal{L}^{-1}[F(s)] = \cos 3t = f(t)$$

So,

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+9} e^{-\frac{\pi s}{2}}\right] = f(t-\tfrac{\pi}{2})u(t-\tfrac{\pi}{2})$$

$$= \cos 3(t-\tfrac{\pi}{2}) u(t-\tfrac{\pi}{2})$$

$$= + \sin 3t \, u(t-\tfrac{\pi}{2})$$

Example: Find the Laplace transform of $t\,u(t-t_0)$.

$$\mathcal{L}[t\,u(t-t_0)] = \mathcal{L}[(t-t_0)u(t-t_0)] + \mathcal{L}[t_0 u(t-t_0)]$$

$$= e^{-st_0}\frac{1}{s^2} + \frac{t_0}{s}e^{-st_0}$$

Example: Find the Laplace transform of

$f(t) = \sin t\, u(t-\pi)$

$f(t) = \sin t\, u(t-\pi) = -\sin(t-\pi)u(t-\pi)$

$F(s) = \mathcal{L}[-\sin(t-\pi)u(t-\pi)] = -\mathcal{L}[\sin(t-\pi)u(t-\pi)]$

$= -\dfrac{1}{s^2+1} e^{-\pi s}$

Example: Solve the differential equation:

$y' + y = f(t)$ where $y(0) = 5$ and

$$f(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 3\sin t & t \geq \pi \end{cases}$$

$f(t) = 3\sin t\, u(t-\pi) = -3\sin(t-\pi)u(t-\pi)$

$F(s) = \mathcal{L}[f(t)] = -3e^{-\pi s} \dfrac{1}{s^2+1}$

$\mathcal{L}[y'+y] = -\dfrac{3}{s^2+1} e^{-\pi s}$

$s\,Y(s) - y(0) + Y(s) = \dfrac{-3}{s^2+1} e^{-\pi s}$

$$Y(s) = \frac{5 + \frac{3}{s^2+1} e^{-\pi s}}{s+1} = \frac{5}{s+1} - \frac{3}{(s^2+1)(s+1)} e^{-\pi s}$$

$$\frac{3}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$A(s^2+1) + (Bs+C)(s+1) = 3$

$A + B = 0, \quad B + C = 0, \quad A + C = 3$

$A = C = \frac{3}{2}$, $B = -\frac{3}{2}$

So,

$$Y(s) = \frac{5}{s+1} + \left[\frac{3/2}{s+1} + \frac{-3/2\, s}{s^2+1} + \frac{3/2}{s^2+1} \right] e^{-\pi s}$$

$$y(t) = 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} - \cos(t-\pi) + \sin(t-\pi)\right] u(t-\pi)$$

$$y(t) = 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \cos t - \sin t\right] u(t-\pi)$$

Multiplication by t^n.

Let's find what happens when we take derivative of $F(s)$ with respect to s.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{dF(s)}{ds} = \int_0^\infty -t e^{-st} f(t)\,dt = -\int_0^\infty t f(t) e^{-st}\,dt$$

$$= -\mathcal{L}[t f(t)]$$

So the Laplace transform of $t f(t)$ is

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$$

Now, taking the second derivative of $F(s)$,

$$\frac{d^2}{ds^2} F(s) = +\int_0^\infty t^2 f(t) e^{-st}\,dt = \mathcal{L}[t^2 f(t)]$$

In general,

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example: Find the Laplace transform of

a) $t \sin \omega t$

b) $t e^{3t}$

c) t^3 using just the fact that $\mathcal{L}[1] = \frac{1}{s}$

Solution:

a) $\mathcal{L}[t \sin \omega t] = -\frac{d}{ds} \mathcal{L}[\sin \omega t]$

$$= \frac{-d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{s^2+\omega^2}$$

b) $\mathcal{L}[t e^{3t}] = -\frac{d}{ds} \mathcal{L}[e^{3t}]$

$$= -\frac{d}{ds} \frac{1}{s-3} = \frac{1}{(s-3)^2}$$

c) $\mathcal{L}[t^3] = (-1)^3 \frac{d^3}{ds^3} \mathcal{L}[1] = (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s}\right]$

$$= (-1) \frac{-6}{s^4} = \frac{6}{s^4}$$

Example: Solve the differential equation

$$y'' + 16y = \cos 4t \qquad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}[y'' + 16y] = \mathcal{L}[\cos 4t]$$

$$s^2 Y(s) - s y(0) - y'(0) + 16\,Y(s) - 16 y(0) = \frac{s}{s^2+16}$$

$$(s^2+16)Y(s) = 1 + \frac{s}{s^2+16}$$

$$Y(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}$$

$$\frac{4}{s^2+16} = \mathcal{L}[\sin 4t]$$

So

$$\frac{1}{s^2+16} = \mathcal{L}\left[\tfrac{1}{4}\sin 4t\right] \quad \text{for the first term}$$

$$\frac{d}{ds}\left[\frac{4}{s^2+16}\right] = \frac{-8s}{(s^2+16)^2}$$

So

$$\frac{s}{(s^2+16)^2} = -\frac{1}{8}\frac{d}{ds}\left[\frac{4}{s^2+16}\right] = \frac{1}{8}\mathcal{L}[t \sin 4t]$$

Therefore,

$$y(t) = \frac{1}{4}\sin 4t + \frac{1}{8} t \sin 4t$$

Laplace transform of Convolution:

Recall that the convolution of two functions $x(t)$ and $h(t)$ is given as

$$y(t) = x(t) * h(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

Theorem:

$$Y(s) = \mathcal{L}[y(t)] = \mathcal{L}[x(t)*h(t)] = X(s)H(s).$$

That is, Laplace transform changes the convolution (in t-domain) into multiplication is s-domain.

Example: Evaluate

$$\mathcal{L}\left[\int_0^t e^\tau \sin(t-\tau) d\tau\right]$$

$$\mathcal{L}\left[\int_0^t e^\tau \sin(t-\tau) d\tau\right] = \mathcal{L}[e^t]\,\mathcal{L}[\sin t]$$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Transform of integral of $f(t)$. Find:

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right]$$

let $g(t) = 1$. Then

$$\int_0^t f(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$$

$$= \mathcal{L}[f(t)]\,\mathcal{L}[1]$$

$$= \frac{F(s)}{s}$$

Similarly

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]$$

Ex: Find the inverse Laplace transform of

$\frac{1}{s(s^2+1)}$ without using partial fractions

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \int_0^t \sin\tau\,d\tau = 1 - \cos t$$

Laplace transform of <u>periodic</u> functions:

We say that $f(t)$ is periodic with period T if $f(t+T) = f(t)$ for all t. That is, if $f(t)$ repeats itself every T seconds.

Now, let's find the Laplace transform of a periodic function $f(t)$.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt$$

Take the second term:

$$\int_T^\infty e^{-st} f(t)\, dt$$

Let $\tau = t - T \Rightarrow t = \tau + T$ and $dt = d\tau$.

Then,

$$\int_T^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-s(\tau+T)} f(\tau+T)\, d\tau$$

$$= e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= e^{-sT} \mathcal{L}[f(t)]$$

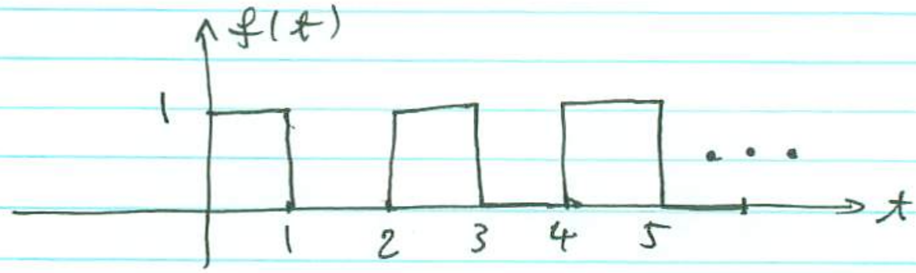
So,

$$\mathcal{L}[f(t)] = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \mathcal{L}[f(t)]$$

From the above, we get,

$$\boxed{\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt}$$

Example: Find the Laplace transform of



Note that this is a periodic function with period $T = 2$.

$$\int_0^T f(t) e^{-st} = \int_0^2 f(t) e^{-st} dt$$

$$= \int_0^1 e^{-st} dt = \frac{1-e^{-s}}{s}$$

$$\boxed{\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}} \frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{s})}}$$