x Lecture 2, July 7, 2006

Sets of points in complex plane.

Take two points at $z = x + iy$ and

$z_0 = x_0 + iy_0$

The difference between these two is

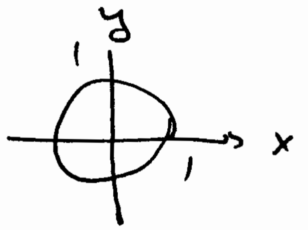
$$z - z_0 = (x - x_0) + i(y - y_0)$$

$$|z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

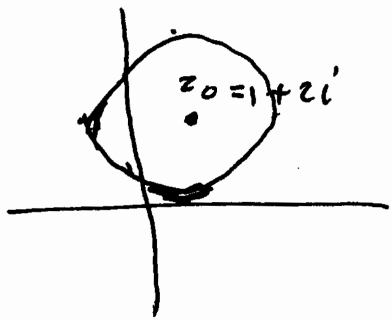
if we let $|z - z_0| = \rho$

Then each z satisfying the above is a point on a circle of radius ρ and center z_0

Example: $|z| = 1$



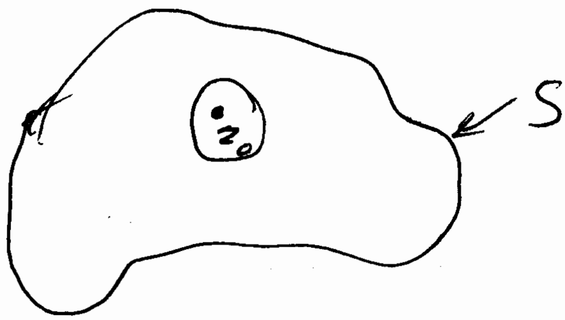
Example $|z - 1 - 2i| = 2$



The points satisfying $|z-z_0|<\rho \quad \rho>0$ are the points ~~lying~~ that lie inside a circle of radius ρ centered at z_0.

This is called a neighborhood or an open disk.

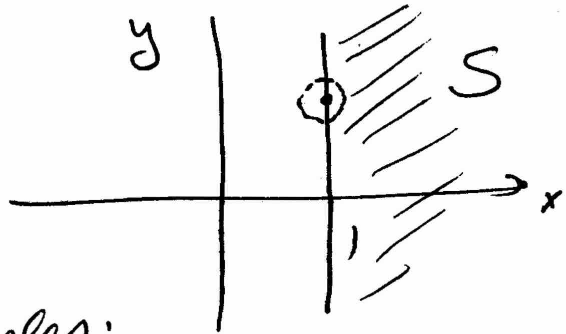
If we have a set S of complex numbers we call z_0 an interior point of S if there is som neighborhood of z_0 that lies entirely in S.



If every point of a set S is an interior point of S then S is said to be an open set.

Example $\mathrm{Re}(z)>1$ is the set of all points $z=x+iy$ with $x>1$. It is open. If we are given $z_0=1.1+2i$ we can have neighborhood $|z-(1.1+2i)|<0.05$ that is entirely in S.

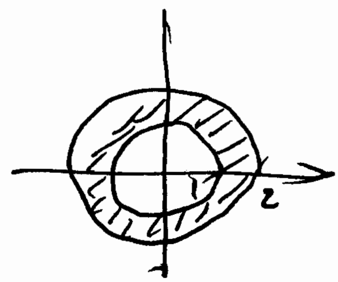
But $Re(z) \geq 1$ is <u>not</u> <u>open</u> since any point on $x=1$ has a neighborhood that is not entirely in S.



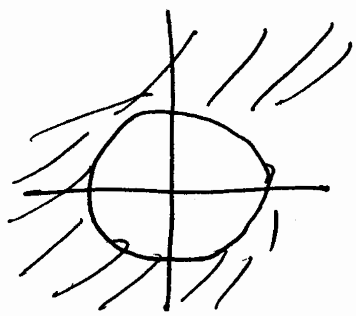
Examples:
The set

$$\rho_1 < |z - z_0| < \rho_2$$

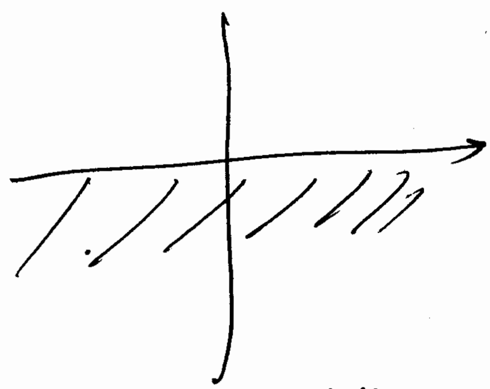
is an open ~~an~~ annulus



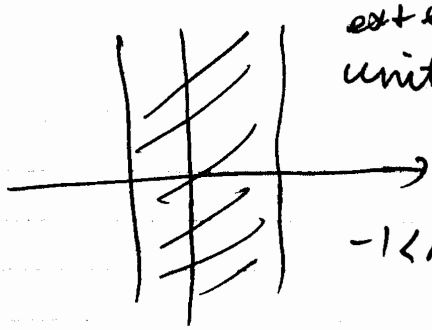
$1 < |z| < 2$



$|z| > 1$
exterior of unit circle



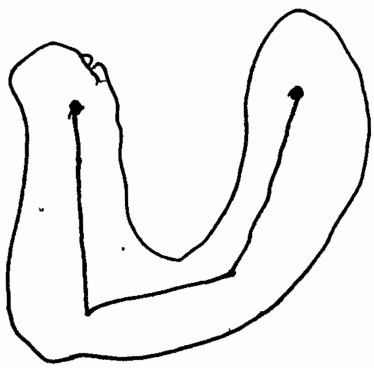
$Im(z) < 0$
lower half plane



$-1 < Re(z) < 1$ infinite strip

If every neighborhood of z_0 has at least one point in S and at least one point outside S, it is called a boundary point of S.

If any two points of S can be connected by a polygonal line ~~the~~ that is entirely in S then S is called connected.



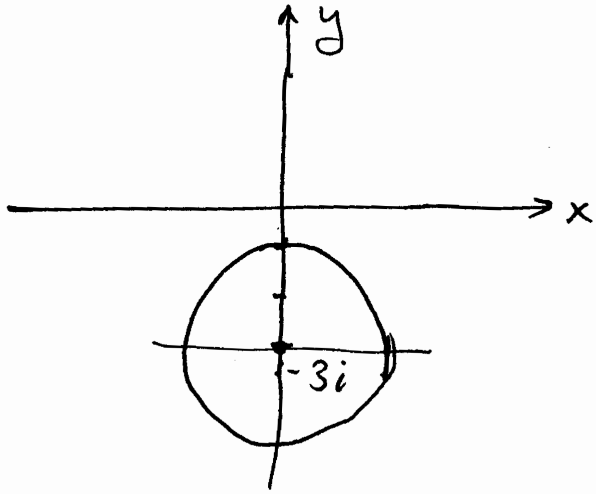
An open connected set is called a domain.

A region is a domain in the complex plane with all, some or none of its boundary points.

Since an open set ~~is~~ has no boundary points it is automatically a domain.

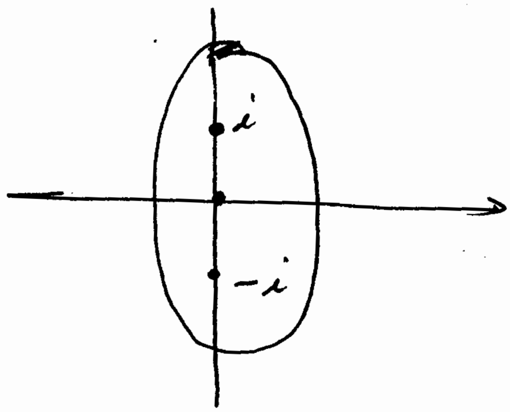
Any set that contains all its boundary points is ~~c~~ called closed.

Example: sketch $|z-3i| = 2$



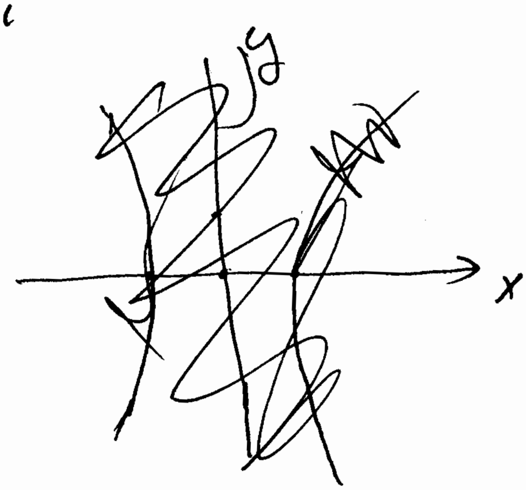
Example: Describe the points satisfying

$$|z-i| + |z+i| = 1$$



an elypse with foci at $\pm i$

Example: Describe



$$z^2 + \bar{z}^2 = 2$$

$$(x+iy)^2 + (x-iy)^2 = 2$$

$$x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2$$

$x^2 - y^2 = 1$ it is a hyperbola

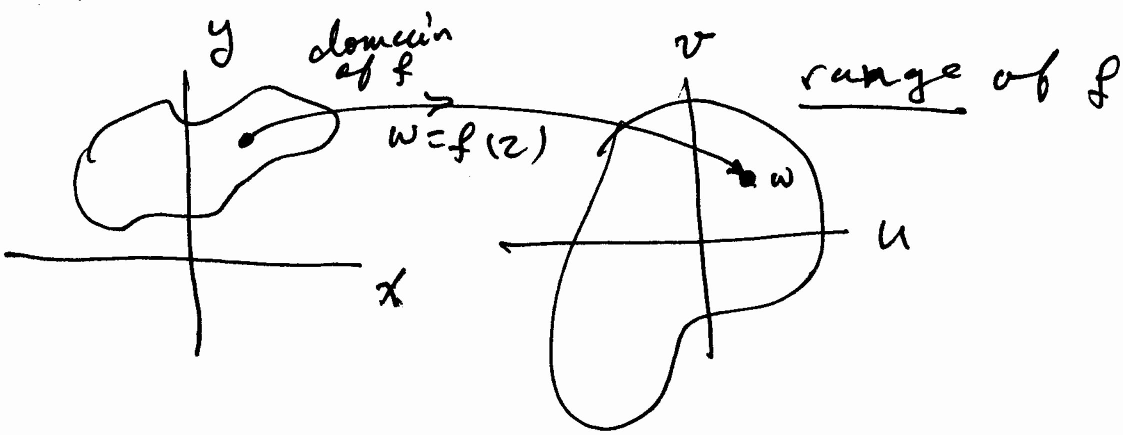
Functions of Complex variables, analyticity

for real variables a function is rule of assigning ~~x to y = f(x) where x~~ members of a set A say $a \in A$ to member of B $(b \in B)$ as $b = f(a)$ where A is called domain and b is the image of a.

in Complex field

$$w = f(z) = u(x,y) + i\, v(x,y)$$

is a function of the complex variable z.
It is a mapping between z-plane and w-plane



Examples:

$f(z) = z^2 - 4z$ $\quad$ z any complex number

$f(z) = \dfrac{z}{z^2+1}$ $\quad$ $z \neq i$ $\quad$ $z \neq -i$

Example: Find the image of the line

$Re(z) = 1$ under mapping $f(z) = z^2$

let $z = x + iy$

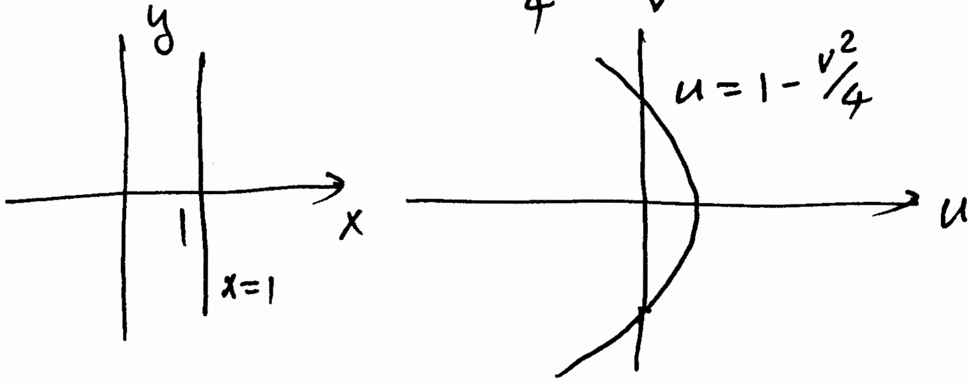
then
$$z^2 = x^2 - y^2 + 2ixy = u + iv$$

or
$$u = x^2 - y^2$$
$$v = 2xy$$

Since $Re(z) = x = 1$ we have

$$u = 1 - y^2 \quad \text{and} \quad v = 2y$$

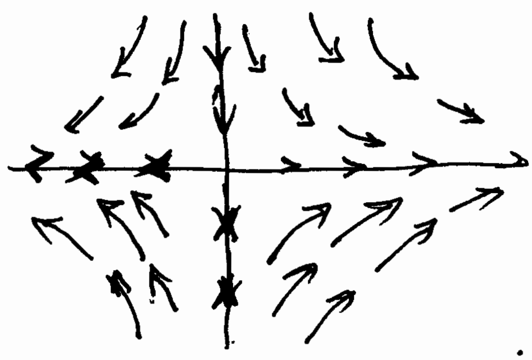
or $u = 1 - \dfrac{v^2}{4}$



Complex functions as flow

We can view $w = f(z)$ as a 2-D fluid flow where ~~flows~~ the vector $f(z)$ specifies the speed and direction of flow at a given point z.

Example: $f(z) = \bar{z} = x - iy$



If $x(t) + iy(t)$ ~~represents~~ is a parametric representation of the path of a particle in the flow then the tangent vector $T = x'(t) + iy'(t)$ must coincide with $f(x(t) + iy(t))$. When $f(z) = u(x,y) + i(x,y)$

$$\frac{dx}{dt} = u(x,y)$$

$$\frac{dy}{dt} = v(x,y)$$

Example:

$$f_3(z) = x - iy$$

then $$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -y$$

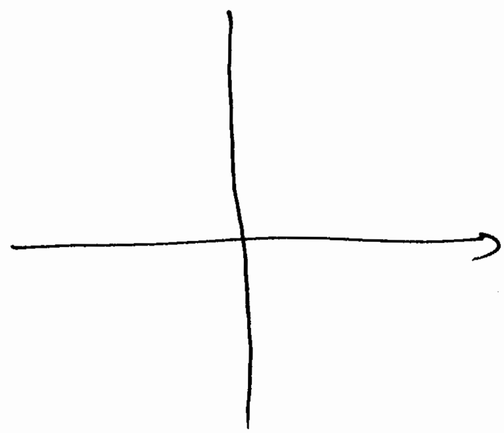
$x(t) = c_1 e^{t}$ and $y(t) = c_2 e^{-t}$

$xy = c_1 c_2$ so points of $x(t) + iy(t)$ lie on the hyperbola $xy = c_1 c_2$

Example $f(z) = z^2$

$$f(z) = (x^2 - y^2) + i\,2xy$$

$$\frac{dx}{dt} = x^2 - y^2$$



$$\frac{dy}{dt} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

the solution of this homogeneous diff. Eq. is $x^2 + y^2 = c_2 y$

Limit of a function

$$\lim_{z \to z_0} f(z) = L$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ if

$$0 < |z - z_0| < \delta$$

Limit of Sum / Product / Quotient

if $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$

then

$$\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$$

$$\lim_{z \to z_0} [f(z) g(z)] = L_1 L_2$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2} \quad \text{if } L_2 \neq 0$$

Continuity: A function is continuous if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Consequence: (of theorem on limit of sum (product)...)

we have

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 \qquad a_n \neq 0$

i.e., any polynomial is continuous since

$\lim_{z \to z_0} z = z_0$

and any rational ~~number~~ function is continuous

except where $h(z)$ (denominator is 0)

$$f(z) = \frac{g(z)}{h(z)}$$

Derivative:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

if $f(z)$ is differentiable at z_0 then it is continuous at z_0

rules of differentiation:

Constant $\frac{d}{dz} c = 0 \qquad \frac{d}{dz} c f(z) = c f'(z)$

Sum $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$

Product rule $\frac{d}{dz}[f(z) g(z)] = f'(z) g(z) + g'(z) f(z)$

Quotient rule $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z) f'(z) - f(z) g'(z)}{[g(z)]^2}$

Chain rule $\cancel{\frac{d}{dz} f(g(z))}$

$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$

Example:

$$f(z) = 3z^4 - 5z^3 + 2z$$

$$f'(z) = 12z^3 - 15z^2 + 2$$

Example: $f(z) = \dfrac{z^3}{4z+1}$

$$f'(z) = \frac{(4z+1)\,2z - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2+2z}{(4z+1)^2}$$

Example: $f(z) = x + 4iy$

$$f(z+\Delta z) - f(z) = (x+\Delta x) + 4i(y+\Delta y) - x - 4iy$$

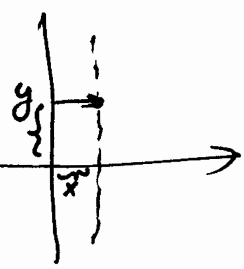
$$= \Delta x + 4i\,\Delta y$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\,\Delta y}{\Delta x + i\,\Delta y}$$

Now if we let $\Delta z \to 0$ along the line parallel to x-axis, $\Delta y = 0 \Rightarrow$ limit $\to 1$

if we let $\Delta x = 0 \Rightarrow$ limit $\to 4$

So, $f(z) = x + 4iy$ is not differentiable at any point.



So, in order for a function to be differentiable

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

~~so~~ much approach the same complex number from ~~along~~ any direction.

Analytic functions

a function $w=f(z)$ is analytic at a point z_0 if f is differentiable at z_0 and at any point in some neighborhood of z_0.

~~Example~~