17.5 Cauchy-Riemann Equations

Can anyone tell me what an analytic function is?

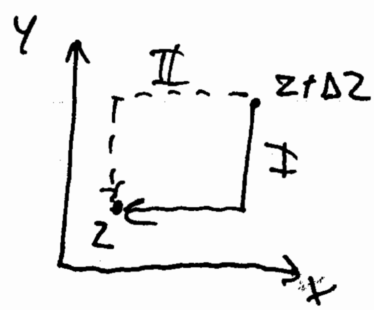
- a function f of a complex variable z is analytic in the domain D if f is differentiable at all points in D.

Why would it not be differentiable?

- for example if we have $f(z) = \bar{z} = x - iy$

we have the derivative at z_0 as

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



path I we set $\Delta y = 0 \Rightarrow 1$

II $\Delta x = 0 \Rightarrow -1$

NOT DIFFERENTIABLE

Cauchy - Riemann equations as a test for ~~differentiability~~. analyticity of a function.

THEOREM If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D, the first partial derivatives exist and satisfy

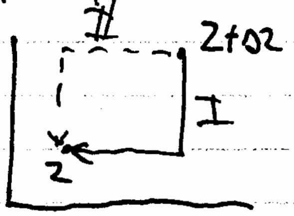
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{AND} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

the official statement as written in the book is

"Let $f(z)$ be defined and continuous in some neighbourhood of a point $z = x + iy$ and differentiable at z itself, then at that point, the 1st order partial derivative of u and v exist and satisfy the Cauchy-Riemann equations.

Hence if $f(z)$ is analytic in a domain D, those partial derivative exist and satisfy the equations at all points of D"

Proof:



$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Use $\Delta z = \Delta x + i\,\Delta y$

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$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Choose Path I $\Delta y \to 0$ first, then $\Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Path II $\Delta x \to 0$ then $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the two we have Cauchy-Riemann Eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{AND} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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Examples

(1) We know $f(z)=z^2$ is analytic for all z, therefore the Cauchy-Riemann equation must be satisfied.

- We have $f(z)=z^2=(x+iy)^2=\underbrace{x^2-y^2}_{u}+i\,\underbrace{(2xy)}_{v}$

$\frac{\partial u}{\partial x}=2x \quad \frac{\partial v}{\partial y}=2x$ ✓

$\frac{\partial u}{\partial y}=-2y \quad \frac{\partial v}{\partial x}=2y$

Theorem 2: A more involved theorem states that the converse of theorem 1 is true.

— In other words: If a real-valued continuous function $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z)=u+i$ is analytic in D.

Example (2) Is $f(z)=z^3$ analytic? $f(z)=(x+iy)^3=\underbrace{x^3-3xy^2}_{u}+i\underbrace{(3x^2y-}_{v}$

$\frac{\partial u}{\partial x}=3x^2-3y^2 \quad \frac{\partial v}{\partial y}=3x^2-3y^2$

$\frac{\partial u}{\partial y}=-6xy \quad \frac{\partial v}{\partial x}=6xy$ ✓ It is analytic

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Example 3 Determination of an analytic function with only 1 part given:

Find the most general analytic function $f(z)$ whose real part is $u(x,y) = x^2 - y^2 - x$

1) $\frac{\partial u}{\partial x} = 2x - 1 = \frac{\partial v}{\partial y} \Rightarrow v = 2xy - y + k(x)$

2) $-\frac{\partial v}{\partial x} = -2y - \frac{dk(x)}{dx}$

and $\frac{\partial u}{\partial y} = -2y \Rightarrow \frac{dk(x)}{dx} = 0 \quad k(x) = \text{real constant}$

$$f(z) = x^2 - y^2 - x + i(2xy - y + k)$$

or $f(z) = z^2 - z + ik$

Harmonic functions

As we just saw, the real and imaginary parts of an analytic function cannot be chosen arbitrarily. In fact, the real and imaginary part of an analytic function satisfy the most important differential equation of physics: Laplace's Equation. (which occurs in gravitation, electrostatics, fluid flow, heat conduction and so on...)

It is this link That makes complex variable so essential to engineering.

Definition: ~~Any~~ A function that ~~solves~~ satisfies Laplace's equation ~~and~~ and has continuous second-order partial derivatives in a domain is said to be harmonic in D.

Laplace's Equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

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~~Proof~~:

Definition 2! If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's equation respectively in D, and have continuous second partial derivatives in D.

THEREFORE if $f(z)$ is analytic in D, the functions u and v are harmonic.

PROOF:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &\overset{\frac{\partial}{\partial x}}{\Longrightarrow} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\overset{\frac{\partial}{\partial x}}{\Longrightarrow} \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \end{aligned}\right\} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

mixed are equal due to continuity.

for v do $\frac{\partial}{\partial y}$ in

Conjugate Harmonic Function

If we have $u(x,y)$ is a harmonic function, it is sometimes possible to find $v(x,y)$ that is also harmonic in D where $f(z) = u(x,y) + iv(x,y)$ is analytic.

Then $v(x,y)$ is a conjugate harmonic fn. of $u(x,$

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example. $u = x^2 - y^2 - y$

1) Verify that it is harmonic

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial^2 u}{\partial x^2} = 2 \quad \frac{\partial u}{\partial y} = -2y-1 \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \checkmark$$

2) Find the conjugate fn.

using Cauchy Riemann

i) $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \Longrightarrow v = 2xy + h(x)$

ii) $-\frac{\partial u}{\partial y} = 2y+1$

$\qquad\qquad \|$

$\frac{\partial v}{\partial x} = 2y + \frac{dh(x)}{dx} \Longrightarrow \frac{dh}{dx} = 1$

$h(x) = x +$

$$v = 2xy + x + C$$

17.6 Exponential and Logarithmic Functions

Now we enter a discussion of elementary complex functions.

We define these so that when z is a purely real number (i.e. x) they reduce to the ~~from~~ functions we are familiar with in calculus.

Exponential function

Recall that $f(x)=e^x$ has properties

$$f'(x)=f(x) \quad \text{and} \quad f(x_1+x_2)=f(x_1)f(x_2)$$

- using Euler's formula we have

$$e^{iy}=\cos y+i\sin y$$

Now for $z=x+iy$ we want $e^{x+iy}=e^x e^{iy}$

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$

Let's see if it satisfies the 3 motivating factors

1) e^z should reduce to familiar function for $z=x$ real
2) e^z is an entire function (analytic for all z)
3) $(e^z)'=e^z$

1) for $z = x$ real we have $y = 0$

$$e^z = e^x(1 + 0i) = e^x$$

2) using Cauchy-Riemann equations

$$u = e^x \cos y \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

3) from The proof of Cauchy-Riemann we used

$f'(z) = u_x + i v_x$ (when we set $\Delta y \to 0$ then $\Delta x \to 0$)

$$(e^z)' = \frac{\partial (e^x \cos y)}{\partial x} + i \frac{\partial (e^x \sin y)}{\partial x} = \cancel{e^x \cos y + i e^x \cos y}$$

$$= e^x \cos y + i e^x \sin y = e^z$$

example ① $z = \frac{\pi i}{2}$ $e^z = ?$

$$e^{\frac{\pi i}{2}} = e^0(\cos \pi/2 + i \sin \pi/2) = i$$

② $$e^{1.4-0.6i} = e^{1.4}(\cos(0.6) - i \sin(0.6)) = 4.055(0.825 - 0.565i)$$
$$= 3.347 - 2.290i$$

③ $z_1 = 2+i$ $z_2 = 4-i$ find $e^{z_1}e^{z_2}$

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2)$$
$$= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)]$$
$$= e^{x_1+x_2}[\cos(y_1+y_2) + i \sin(y_1+y_2)]$$
$$= e^{z_1+z_2}$$
$$= e^6$$

Note that $e^z = e^x e^{iy} \neq 0$ since $e^x \neq 0$

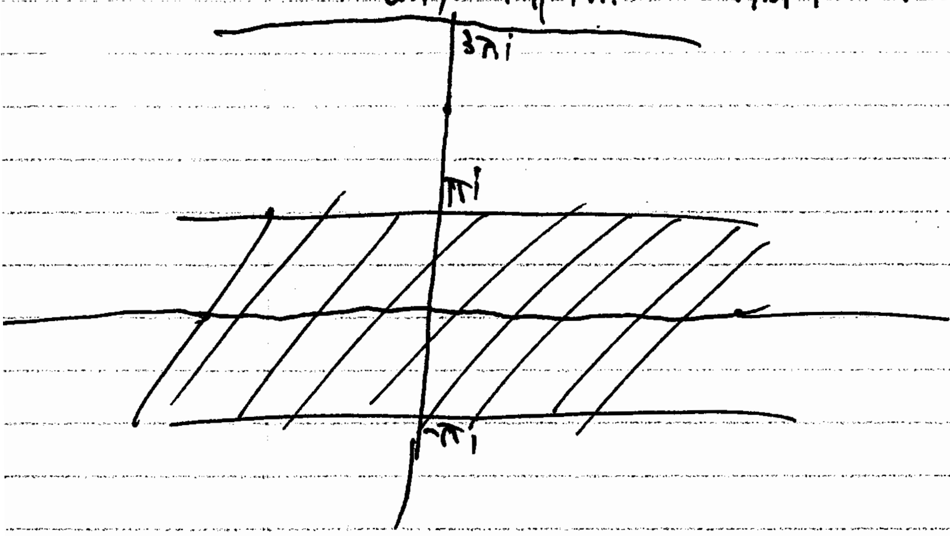
Periodicity:

since $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

we see that e^z is periodic with complex period 2π

$$\boxed{e^z = e^{z+2\pi i} \quad \text{for all } z}$$

Because of this, all possible values of e^z are assumed in any infinite horizontal strip of width 2π.



So for any point z in the strip $-\pi < y < \pi$ values of $f(z+2\pi i)$, $f(z+4\pi i)$ ... are the same

The strip $-\pi < y \le \pi$ is called the <u>fundamental strip</u>

Polar form:

You previously saw that in polar form z can be written as

$$z = r(\cos\theta + i\sin\theta)$$

We can therefore rewrite it as

$$z = r e^{i\theta}$$

example:

$$z = 1 + i \qquad r = |z| = \sqrt{2} \qquad \theta = \pi/4$$

$$z = \sqrt{2}\, e^{\pi/4}$$

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Logarithmic function

the natural logarithm of $z = x+iy$ is denoted a $\ln z$ and is the inverse of the exponential function.

$w = \ln z$ for $z \neq 0$ is defined by $e^w = z$

$W = u + iv$

We have
$$
\begin{aligned}
z &= e^w \\
x+iy &= e^{u+iv} \\
&= e^u(\cos v + i \sin v) \\
re^{i\theta} &= e^u \cos v + i e^u \sin v
\end{aligned}
$$

giving $e^u = r$ $\quad u = \ln r$ $\quad$ where $r = |z| > 0$

$v = \theta$ $\quad\quad \theta = \arg z$

$$\boxed{\ln z = \ln r + i\theta}$$

Note: the argument of z is determined up to integer multiples of 2π, $\therefore$ the complex natural logarithm is infinitely many-valued.

THIS makes sense since the exponential fn. is periodic

$$\boxed{\ln z = \ln|z| + i(\theta + 2n\pi) \quad n = 0, \pm 1, \pm 2, \ldots}$$

We have the principal value of $\ln z$ which correspond to the principal argument of z and is given by (using $n=0$)

$$\operatorname{Ln} z = \ln|z| + i \underbrace{\operatorname{Arg} z}_{\text{principal argument of } z}$$

$$-\pi < \operatorname{Arg} z \leq \pi$$

Example: $z = 3-4i \quad \ln(3-4i) = ?$

$|z| = 5 \quad \theta = \operatorname{Arg} z = -0.927$

$$\ln(3-4i) = \ln 5 + i(-0.927 + 2n\pi) \qquad n = 0, \pm 1, \pm 2$$

$$= 1.609 - 0.927i + 2n\pi$$

$$\operatorname{Ln}(3-4i) = 1.609 - 0.927i$$

Note that

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

$$\text{and } \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

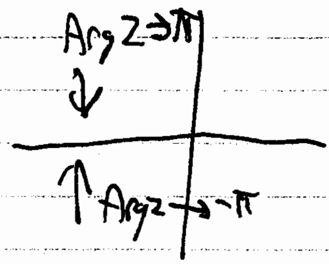
still holds. But they hold in the sense that each value of one side is contained among the values of the other side. Not necessarily true if we use the principal values.

Analyticity of Logarithm $\operatorname{Ln}(z)$

1) Each value of $\ln(z)$ represents a branch. Each is analytic, we show ~~[illegible]~~ for Ln

2) $\operatorname{Ln}(0) = f(0)$ does not exist.

3) And $f(z)$ is discontinuous at any point on negative x-axis



We can select D as the domain that doesn't include the negative axis ⇒ BRANCH CUT

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1) $\ln z = u + iv$

2) $z = x + iy \qquad |z| = \sqrt{x^2+y^2}$

3) $u = \ln|z| = \frac{1}{2}\ln\left(x^2+y^2\right)$

$v = \tan^{-1}\dfrac{y}{x}$

Note:

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\,2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+y^2/x^2}\,\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2/x^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

✓

Derivative $\quad (\mathrm{Ln}\, z)' = \underbrace{u_x + i v_x}_{\text{used in Cauchy Riemann eqn.}} = \dfrac{x}{x^2+y^2} + i\,\dfrac{-y}{x^2+y^2} = \dfrac{x-iy}{x^2+y^2} = 1/$

$$\boxed{\frac{d}{dz}\mathrm{Ln}\, z = 1/z}$$

Complex Powers

In real variables we have $x^a = e^{a \ln x}$

If $z = x + iy$ is complex then z^α is defined as

$$\boxed{z^\alpha = e^{\alpha \ln z} \qquad z \neq 0}$$

and α is complex

- since $\ln z$ is multivalued, z^α is generally also multivalued.

if α is an integer we have only one value

$$z^k = e^{k \ln z} = e^{k(\ln|z| + i(\theta + 2\pi n))} = e^{k \ln|z| + i(\theta + 2\pi n k)}$$

all values of n give same result

if $k = 1/p$ we have the usual p^{th} root with p distinct values.

Example: $z = i \quad \alpha = 2i$ find z^α

$$z^\alpha = i^{2i} = e^{2i \ln i} = e^{2i[\ln 1 + i(\pi/2 + 2n\pi)]}$$

$$= e^{-(1+4n)\pi} \qquad n = 0, \pm 1, \pm 2, \pm 3 \ldots$$