(I)

17.7 Trigonometric functions
Hyperbolic functions

Just as we extended e^{x} to e^{z}, we want complex Trigonometric functions that extend the familiar real Trigonometric functions.

Remember Euler's equations

even fn, odd fn.

(1) $e^{ix} = \cos x + i\sin x$, (2) $e^{-ix} = \cos x - i\sin x$

Add (1)+(2) $\Rightarrow 2\cos x \Rightarrow \cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

Subtract (1)-(2) $\Rightarrow 2i\sin x \Rightarrow \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$\therefore$ for $z = x+iy$

we have
$$\boxed{\begin{aligned}\cos z &= \tfrac{1}{2}\left(e^{iz}+e^{-iz}\right)\\ \sin z &= \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)\end{aligned}}$$

II

As with real variable, we have

$\tan z = \dfrac{\sin z}{\cos z}$ $\quad \cot z = \dfrac{1}{\tan z}$ $\quad \sec z = \dfrac{1}{\cos z}$ $\quad \csc z = \dfrac{1}{\sin z}$

Analyticity

We know from before that e^{iz} and e^{-iz} are entire functions.

Therefore $\sin z$ and $\cos z$ are also entire function

We know from Trigonometry that $\sin(n\pi)=0$ for $n=0,\pm1,$

And $\quad \cos\left(\dfrac{(2n+1)\pi}{2}\right)=0$ for $n=0,\pm1,$

Therefore $\tan z$ and $\sec z$ are analytic at all points except for $z=(2n+1)\pi/2$

And $\cot z$ and $\csc z$ are analytic at all points except for $z=n\pi$

III

Derivatives

Using $\frac{d}{dz} e^z = e^z$ we have $\frac{d}{dz} e^{iz} = i e^{iz}$

and $\frac{d}{dz} e^{-iz} = -i e^{-iz}$

therefore $\frac{d}{dz} \cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz}-ie^{-iz}}{2}$ $\boxed{i = \frac{-1}{i}}$

$= -\frac{[e^{iz}-e^{-iz}]}{2i} = -\sin z$

We can also obtain the following

$\frac{d}{dz}\sin z = \cos z$ $\qquad$ $\frac{d}{dz}\cos z = -\sin z$

$\frac{d}{dz}\tan z = \sec^2 z$ $\qquad$ $\frac{d}{dz}\cot z = -\csc^2 z$

$\frac{d}{dz}\sec z = \sec z \tan z$ $\qquad$ $\frac{d}{dz}\csc z = -\csc z \cot z$

(for more complicated use $\frac{d}{dx}\left(\frac{M}{N}\right) = \frac{\frac{d}{dx}M \cdot N - \frac{d}{dx}N \cdot M}{N^2}$)

IV

We can see from

$$\sin z = \frac{e^{iz} \cancel{+} e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{iz} - e^{-iz} = 2i \sin z \qquad 2\cos z = e^{iz} + e^{-iz}$$

$$2\cos z + 2i \sin z = 2e^{iz}$$

$$\boxed{e^{iz} = \cos z + i \sin z}$$

Euler's formula is indeed valid in complex.

The familiar trig identities are still valid in the complex case

odd fn. even fn

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2(z) + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$$

Next we ~~can~~ can write the hyperbolic sin and cosine defined in term of e^{y} and e^{-y} where y is real

$$\boxed{\sinh = \frac{e^{y}-e^{-y}}{2}} \quad \text{and} \quad \boxed{\cosh y = \frac{e^{y}+e^{-y}}{2}}$$

So we have $\cos z = \frac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right]$

$$= \frac{1}{2}\left[e^{-y}e^{ix} + e^{y}e^{-ix}\right]$$

from $-x$

$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$

$$= \frac{1}{2}\left(e^{y}+e^{-y}\right)\cos x - \frac{1}{2}\left(e^{y}-e^{-y}\right)\sin x \; i$$

$$\boxed{\cos z = \cosh y \cos x - i \sinh y \sin x}$$

Similarly we have $\boxed{\sin z = \cosh y \sin x + i \sinh y \cos x}$

using $\cosh^2 y = 1 + \sinh^2 y$ we have

$$|\cos z|^2 = (\cosh y \cos x - i \sinh \sin x)(\cosh y \cos x + i \sinh y \sin x)$$

$$= \cosh^2 y \cos^2 x + \sinh^2 y \sin^2 x$$

VI

$|\cos z|^2 = (1+\sinh^2 y)\cos^2 x + \sinh^2 y \sin^2 x$

$\boxed{|\cos z|^2 = \cos^2 x + \sinh^2 y}$

similarly we have $\boxed{|\sin z|^2 = \sin^2 x + \sinh^2 y}$

Zeros only occur when $|\cdot|^2$ is 0

Therefore for $\cos z$, zeros are when both $\cos^2 x$ and $\sinh^2 y$ are 0.

- Therefore $y=0$ since $|\sinh y| \to \infty$ as $y \to \infty$

and $x = (2n+1)\pi/2$

- For $\sin z$ it is zero as $y=0$ and $x = n\pi$

Also having $|\sinh y| \to \infty$

means that $|\sin z| \leq 1$ is NOT TRUE

while $|\sin x| \leq 1$ <u>is</u> TRUE

VII

example 1

$\cos z = 5$ (which has no real solutions)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 5$$

$$e^{iz} + e^{-iz} = 10 \qquad \times\, e^{iz}$$

$$e^{2iz} - 10e^{iz} + 1 = 0 \qquad \text{using } e^{iz} = \frac{10 \pm \sqrt{96}}{2}$$

$$= 9.899$$

$$\text{or } 0.101$$

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = 9.899 \text{ or } 0.101$$

$$e^{-y} = 9.899 \text{ or } 0.101 \quad \text{and } e^{ix} = 1$$

because there is no imaginary element

$$x = 2n\pi \qquad y = \pm 2.292$$

$$z = 2n\pi \pm 2.292\, i \qquad n = 0, \pm 1, \pm 2, \ldots$$

VIII

Hyperbolic functions

We define the complex hyperbolic sine and cosine as

$$\boxed{\cosh z = \frac{e^{z}+e^{-z}}{2}} \qquad \boxed{\sinh z = \frac{e^{z}-e^{-z}}{2}}$$

Again using $\frac{d}{dz} e^{z} = e^{z}$

we can obtain

$$\frac{d}{dz}\cosh z = \sinh z \qquad \frac{d}{dz}\sinh z = \cosh z$$

And we can define the following functions

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{1}{\tanh z} \qquad \operatorname{sech} z = \frac{1}{\cosh z}$$

$$\operatorname{csch} z = \frac{1}{\sinh z}$$

The hyperbolic sine and cosine are entire fns.

and the fns are analytic except at points where the denominator is 0.

If we replace z by iz we get

$$\boxed{\begin{aligned}\cosh iz &= \cos z\\ \sinh iz &= i \sin z\end{aligned}}$$

AND

$$\boxed{\begin{aligned}\cos iz &= \cosh z\\ \sin iz &= i \sinh z\end{aligned}}$$

So we have functions that are unrelated in the real domain but are related in the complex domain.

Using $\cos z = \cos x \cosh y - i \sin x \sinh y$

we have $\cosh z = \cos iz = \cos(-y+ix)$ $\quad x \to -y$, $y \to x$

$$= \cos(-y) \cosh x - i \sin(-y) \sinh x$$

$$\boxed{\begin{aligned}\cosh z &= \cos y \cosh x + i \sin y \sinh x\\ \sinh z &= \cos y \sinh x + i \sin y \cosh x\end{aligned}}$$

Similarly (for the second line)

X

The zeros are therefore purely imaginary and for $\cosh z$ they are $z = (2n+1)\frac{\pi i}{2}$

for $\sinh z$ $\quad z = n\pi i$

Proof: We obtain them from

$$\sinh z = \cos y \sinh x + i \sin y \cosh x$$

$$|\sinh z|^2 = \cos^2 y \sinh^2 x + \sin^2 y \cosh^2 x$$

$$\cosh^2 x = 1 + \sinh^2 x$$

$$|\sinh z|^2 = \cos^2 y \sinh^2 x + \sin^2 y (1 + \sinh^2 x)$$

$$= \sin^2 y + \sinh^2 x$$

$\sinh z$ zeros at $y = n\pi$

$\cosh z$ zeros at $y = \frac{(2n+1)\pi}{2}$

Using $\sinh z = \cos y \sinh x + i \sin y \cosh x$

$\cosh z = \cosh x \cos y + i \sin y \sinh x$

we see that they are periodic with period $2\pi i$

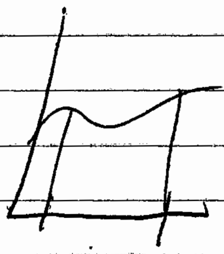
$\sinh z = \sinh(z + 2\pi i n)$

XI

Contour Integrals

In real math we evaluate an indefinite integral over an interval on the real line

e.g.



In the case of complex math,

we integrate along a curve C in the complex plane. We call this the path of integration.

We define a path $z(t) = x(t) + iy(t)$ $a \le t \le b$

for example $z(t) = t + 3it$ $0 \le t \le 2$

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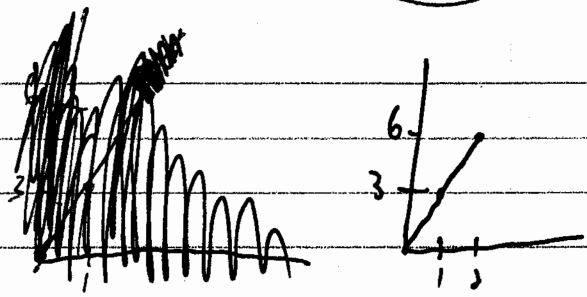
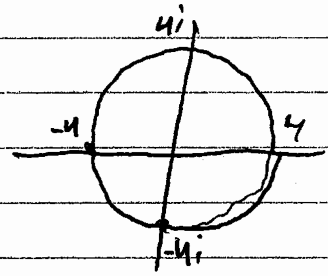
If we plot it



this represents a portion on the line $y = 3x$

We could also have $z(t) = 4\cos t + 4i\sin t \quad (-\pi \leq t \leq \pi$

Graphically

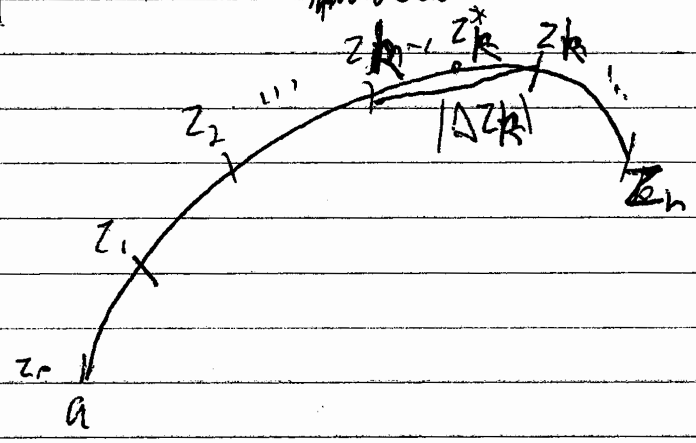


C is called a smooth curve if it has a continuous and nonzero derivative at each point.

$$\frac{dz}{dt} = \frac{dx(t)}{dt} + i\,\frac{dy(t)}{dt}$$

XIII

If we Take a smooth curve



- We can subdivide The curve into n partitions

$t_0 = a < t_1 < \dots < b = t_n$

giving us points

$z_0, z_1, \dots z_n$

- In each partition we select a point z_k^* that falls somewhere on IT. and we let $\Delta z_k = z_k - z_{k-1}$

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We Take the sum $S_n = \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$

If we maximize the number n and select the partitions such that the greatest value of $|\Delta z_k|$ is minimized we obtain the definition of a line integral over the curve C (which is the path of integration).

$$\boxed{\int_C f(z)\,dz = \lim_{\max|\Delta z_k| \to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k}$$

if C is a closed path (meaning it starts and ends at the same point), we write

$$\oint_C f(z)\,dz$$

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- the limit exists if f is continuous on all points on C and C is either smooth or piecewise smooth.

If we set $\Delta Z_k = \Delta X_k + i\Delta Y_k$

and $Z_k^* = \gamma_k + i\eta_k$ $\quad f(Z_k^*) = u + iv$

$u(\gamma_k, \eta_k)$

$v(\gamma_k, \eta_k)$

we can rewrite the sum as

$$S_n = \sum_{k=1}^{n} (u + iv)(\Delta X_k + i\Delta Y_k)$$

$$= \sum u\Delta X_k - \sum v\Delta Y_k + i\left[\sum u\Delta Y_k + \sum v\Delta X_k\right]$$

Remember that we have

$$\int_C f(z)\,dz = \lim_{\max|\Delta Z_k|\to 0} S_n$$

$$= \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]$$

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Since u and v are functions of x and y which in turn are functions of t we can write

$$\int_C f(z)\,dz = \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_a^b\left(u\frac{dy}{dt} + v\frac{dx}{dt}\right)dt$$

$$= \int_a^b \Big(u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\Big)\,dt$$

$$+ i\int_a^b \Big(u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'$$

$$= \int_a^b \Big(u(x(t),y(t)) + i\,v(x(t),y(t))\Big)\Big(x'(t) + i\,y'(t)\Big)\,dt$$

$$\boxed{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$$

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Example 1

$$\oint_C \frac{dz}{z} = ? \qquad C \text{ is unit circle counterclockwise}$$

$$f(z) = 1/z \qquad z(t) = \cos t + i\sin t \quad 0 \le t \le 2\pi$$

the insures counterclockwise

$$f(z(t)) = \frac{1}{\cos t + i\sin t} \qquad z'(t) = -\sin t + i\cos t$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{\cos t + i\sin t}(-\sin t + i\cos t)\,dt$$

$$= \int_0^{2\pi} \frac{i(\cos t + i\sin t)}{\cos t + i\sin t}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Could also be solved using polar form.

$$z(t) = e^{it} \qquad \frac{1}{z(t)} = e^{-it} \qquad z'(t) = +ie^{+it}$$

$$\int_0^{2\pi} \frac{+ie^{+it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Example 2 $m \to$ integer

$f(z) = (z - z_0)^m$ integrate counterclockwise over circle of unit radius $C: |z - z_0| = 1$

C can be represented by

$z(t) = z_0 + \cos t + i \sin t = z_0 + e^{it}$ $0 \le t \le 2\pi$

$$(z - z_0)^m = e^{imt}$$

$$\oint_C (z-z_0)^m \, dz = \int_0^{2\pi} \underbrace{e^{imt}}_{f(z(t))} \; \underbrace{i e^{it}}_{z'(t)} \, dt$$

$$= \int_0^{2\pi} i e^{i(m+1)t} \, dt$$

$$= \int_0^{2\pi} i \left[\cos[(m+1)t] + i \sin[(m+1)t] \right] dt$$

if $m \neq -1$

$$\oint_C (z-z_0)^m \, dz = 0$$

if $m = -1$

$$\oint_C (z-z_0)^m \, dz = \int_0^{2\pi} i \, dt = 2\pi i$$

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Properties of Contour Integrals

Linearity

1) $\int_C k f(z)\,dz = k \int_C f(z)\,dz$ $\quad k$ constant

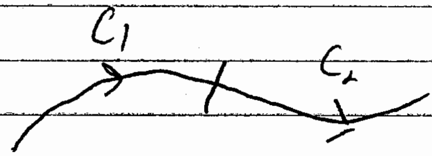
we can remove constants

and

2) superposition applies

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

3) We can decompose the path C into 2 paths



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

4) If we reverse the sense of integration, we get the negative of the original

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$$

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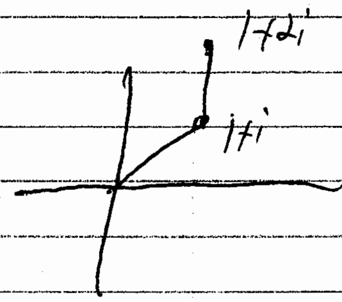
Example



Evaluate $\int_C (x^2+iy^2)\,dz$

$$= \int_{0,\,C_1} (x^2+iy^2)\,dz + \int_{C_2} (x^2+iy^2)\,dz$$

C_1 $\quad y=x \quad x=x \quad 0\le x\le 1 \qquad z(x)=x+ix$

$z'(x)=1+i$

$$\int_{C_1} (x^2+iy^2)\,dz = \int_0^1 (x^2+ix^2)(1+i)\,dx$$

$$= \int_0^1 2ix^2 = \frac{2ix^3}{3}\Big|_0^1 = \frac{2i}{3}$$

C_2 $\quad x=1 \quad y=y \quad 1\le y\le 2$

$z(y)=1+iy \quad z'(y)=i$

$$\int_{C_2} (x^2+iy^2)\,dz = \int_1^2 (1+iy^2)\,i\,dy = \int_1^2 (i-y^2)\,dy$$

$$= iy - y^3/3\Big|_1^2 = 2i - 8/3 - i + 1/3$$

$$= -7/3 + i$$

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$$\int_{C_1} (x^2 + i y^2)\, dz = -7/3 + 5/3\, i$$

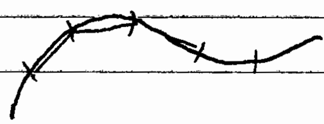
Bound on Absolute Value of a Contour Integral

$$|S_n| = \left| \sum_{k=1}^{n} f(z_k^*)\, \Delta z_k \right| \underbrace{\leq}_{\text{due to the triangle inequality}} \sum_{k=1}^{n} |f(z_k^*)|\, |\Delta z_k|$$

$$\leq M \sum_{k=1}^{n} |\Delta z_k|$$

when $|f(z)| \leq M$ for all C

$$\sum_{k=1}^{n} |\Delta z_k| \leq L$$



In fact As $\Delta z_k \to 0$ $\quad \sum_{k=1}^{n} |\Delta z_k| \to L$ (length of the curve)

So we have as a bound

$$\boxed{\int_C f(z)\, dz \leq ML}$$

L is length of C

$|f(z)| \leq M$

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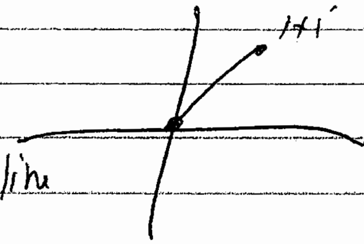
Example:

$$\int_C z^2\,dz$$ C is straight line



$L = \sqrt{2}$ $\quad |f(z)| = |z^2| \le 2$ (at $z = 1+i$)

$\therefore M = 2$

$$\left|\int_C z^2\,dz\right| \le 2\sqrt{2} = 2.8284$$

True value $\quad z(x) = x + ix \quad z'(x) = 1+i$

$z^2 = (x+ix)(x+ix) = 2ix^2$

$$\int_C z^2\,dz = \int_0^1 (2ix^2 - 2x^2)\,dx = \frac{2ix^3}{3} - \frac{2x^3}{3}\Big|_0^1 = -2/3 + 2/3\,i$$

$$|-2/3 + 2/3\,i| = 0.9428$$