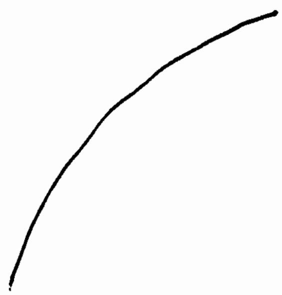
X 5th. lecture, July 18, 2006

Contour Integral



a curve is given in terms of $x = x(t)$ and $y = y(t)$.

e.g.,

Ex. 1) $x(t) = a\cos t$

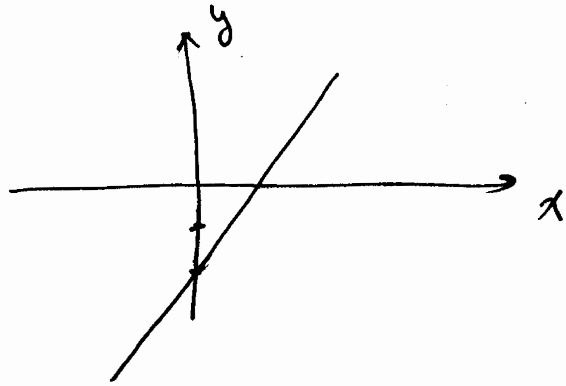
$y(t) = a\sin t$

or $x^2 + y^2 = a^2$, i.e., a circle

or

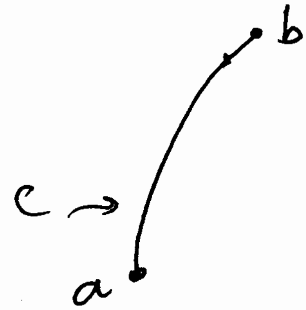
Ex. 2) $x(t) = 2t + 1$

$y(t) = 6t + 1$ $\Rightarrow y = 3x - 2$



Theorem

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$



Example

$$\int_C (z+3)\,dz \quad \text{where } C \text{ is } x = 2t,\ y = 4t-1$$
$$1 \leq t \leq 3$$

$$z(t) = 2t + i(4t-1)$$

$$z'(t) = 2 + i4 \quad , \quad z + 3 = (2t+3) + i(4t-1)$$

$$\int_C (z+3)\,dz = \int_1^3 \left[(2t+3) + i(4t-1)\right](2+i4)\,dt$$

$$= \int_1^3 \left[(-12t+10) + i2(8t+5)\right] dt$$

$$= -2\int_1^3 (6t-5)\,dt + i2\int_1^3 (8t+5)\,dt$$

$$= -2\left[3t^2 - 5t\right]\Big|_1^3 + i2\left[4t^2 + 5t\right]\Big|_1^3$$

$$= -2\left[27 - 15 - 3 + 5\right] + i2\left[36 + 15 - 4 - 5\right]$$

$$= \boxed{-28 + i84}$$

Example:

Find the contour integral of $f(z) = \frac{1}{z}$ along the unit circle, i.e., $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$z = \cos t + i \sin t = e^{it} \Rightarrow z'(t) = i e^{it}$

$$\oint_C f(z)\,dz = \oint_C \frac{1}{z} z'\,dt = \int_0^{2\pi} e^{-it}\, i\, e^{it}\,dt$$

$$= i\int_0^{2\pi} dt = \boxed{2\pi i}$$

Properties of Contour integral

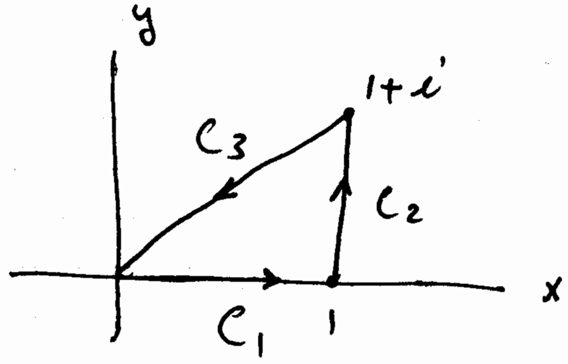
1) $\int_C k f(z)\,dz = k \int_C f(z)\,dz$

2) $\int_C (f(z) + g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

3) $\int_{C_1 \cup C_2} f(z)\,dz = \int_{C} f(z)\,dz + \int_{C_2} f(z)\,dz$

4) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

Example: Find the integral of $f(z) = z^2$ along the curve C given as:



$$\oint_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} dz + \int_{C_3} dz$$

C_1 is given as $x(t) = t$ and $y(t) = 0$

or $z(t) = t$ $\quad 0 \leq t \leq 1$

$$\int_{C_1} z^2 dz = \int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

C_2 is given as $x(t) = 1$, $y(t) = t$

or $z(t) = 1 + it$ $\quad 0 \leq t \leq 1$, $z'(t) = i$

$$\int_{C_2} f(z) dz = \int_0^1 (1+it)^2 i \, dt$$

$$= -2\int_0^1 t \, dt - i\int_0^1 (t^2 - 1) \, dt = -1 + \frac{2}{3} i$$

C_3 is defined as $x(t)=1-t$, $y(t)=1-t$

$z(t)=(1-t)+i(1-t) \qquad 0 \leq t \leq 1$

$z'(t) = -1-i$

$f(z) = (1-\cancel{t})^2 - (1-\cancel{t})^2 + 2i(1-t)^2 = 2i(1-t)^2$

$$\int_{C_3} f(z)\,dz = -2i \int_0^1 (1-t)^2 (1+i)\,dt$$

$$= -2i(1+i)\int_0^1 (1-t)^2\,dt = -\frac{2}{3}(i-1)$$

so

$$\oint_C f(z)\,dz = \frac{1}{3} - 1 + \cancel{\frac{2}{3}i} - \cancel{\frac{2}{3}i} + \frac{2}{3} = 0$$

<u>Theorem: Bounding of an integral</u>

If $|f(z)| \leq M$ then $\left|\int_C f(z)\,dz\right| \leq ML$

where L is the length of contour C.

Example : Find a bound (an upper bound) on the absolute value of $\oint_C \frac{e^z}{z^2+1}\,dz$ where C is the circle $|z|=5$

$$\left|\frac{e^z}{z^2+1}\right| = \frac{|e^z|}{|z^2+1|} \leq \frac{|e^z|}{|z^2|} = \frac{|e^z|}{|z|^2} = \frac{|e^z|}{25}$$

$$|e^z| = |e^x \cdot e^{iy}| = |e^x| \leq |e^5|$$

So $\left|\dfrac{e^z}{z^2+1}\right| \leq \dfrac{e^5}{25} = M$

$L = 2\pi(5) = 10\pi$

Therefore,

$$\boxed{\left|\oint_C \frac{e^z}{z^2+1}\,dz\right| \leq \frac{10\pi e^5}{25}}$$

Simply Connected Domains

A domain is called simply connected if any contour in it can be shrunk to a point without leaving the domain D. In another word, a simply connected domain does not have holes

Simply connected Domain

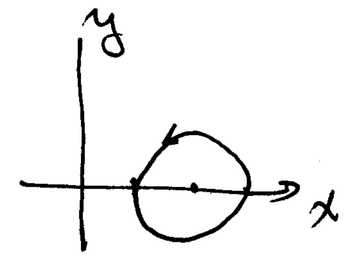
Multiply Connected domain

Cauchy Theorem:

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$

Goursat showed that the assumption of continuity of f' in D is not necessary.

Example: Find the integral of e^z over C given as $|z-2|=1$



$$e^z = e^x \cdot e^{iy} = e^x \cos y + i e^x \sin y$$

$u(x,y) = e^x \cos y$, $\quad v(x,y) = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

So $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

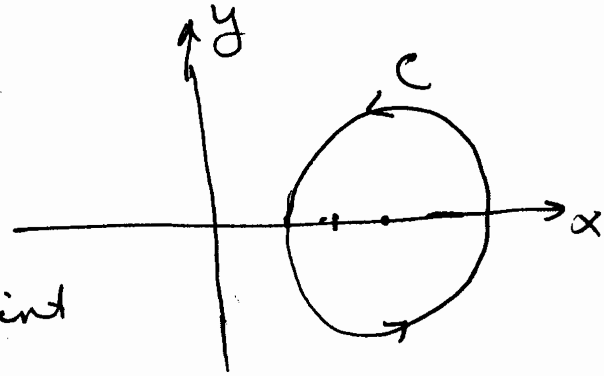
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so e^z is analytic everywhere (it is an entire function) and the contour C is a simple closed curve. So

$$\oint_C e^z dz = 0$$

Example:

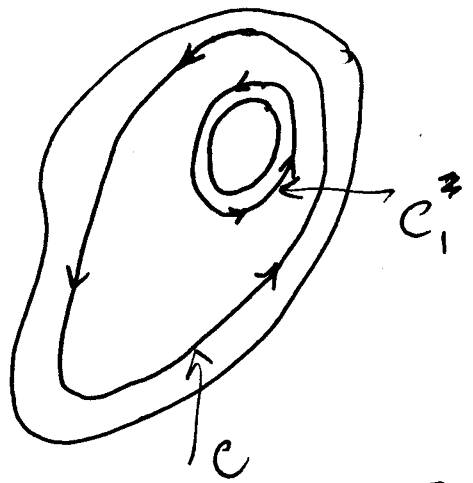
Find $\oint_C \frac{dz}{z^2}$ where C is a circle given by $|z-3| = 2$



$f(z) = \frac{1}{z^2}$ is analytic except at $z = 0$. This point is not inside C so

$$\oint_C \frac{dz}{z^2} = 0$$

Cauchy Theorem for multiply connected Domains

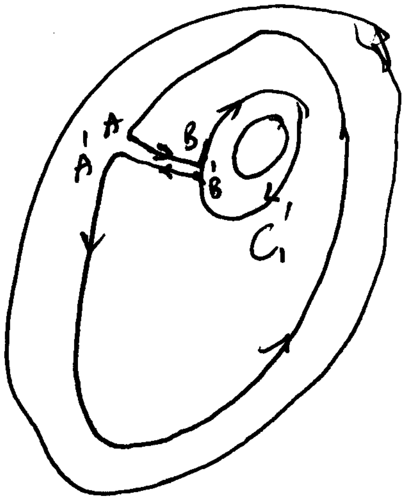


Take a domain with one hole (a doubly connected domain).

Take to simply closed ~~connected~~ curves such as C and C_1.

Suppose f is analytic on C and C_1 and any poin inside C but outside C_1.

Now lets make a cut in the domain as shown by points A, B, B', A'



Then the curves C, AB, C_1' and $B'A$ form a simple closed curve in a simply connected domain (C_1' is C_1 traversed in reverse direction, i.e., clockwise).

So

$$\oint_{C_*} f(z)dz + \int_A^B f(z)dz + \oint_{C_1} f(z)\,dz + \int_{B'}^{A'} f(z)dz = 0$$

But $\displaystyle\int_A^B f(z)\,dz = -\int_{B'}^{A'} f(z)\,dz$

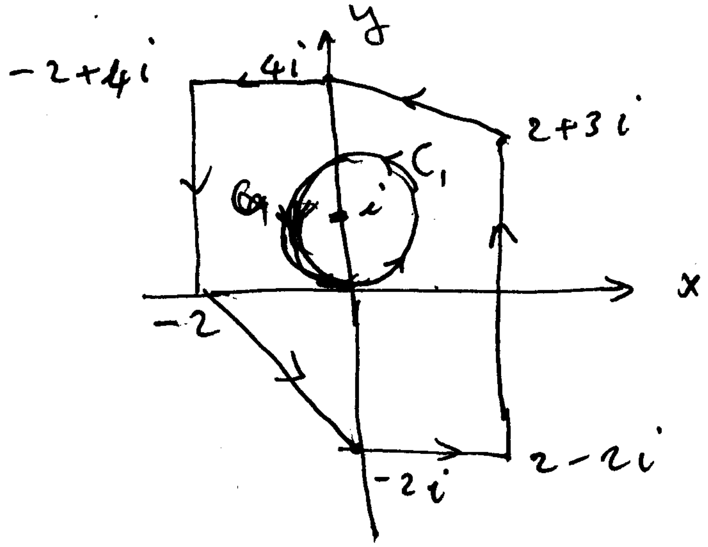
So

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

This is called the principle of <u>deformation</u> of <u>contours</u>.

Example: Evaluate

$\displaystyle\oint_C \frac{dz}{z-i}$ ~~over~~ where C is the outer contour shown below



Consider C_1 given by $|z-i| = 1$

let $x = \cos t$ and $y = 1 + \sin t$ $\qquad 0 \le t \le 2\pi$

or $z = i + e^{it}$ $\qquad 0 \le t \le 2\pi$

Then $z - i = e^{it}$ and $z' = ie^{it}$, so

$$\oint_C \frac{dz}{z-i} = \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} = i\int_0^{2\pi} dt = \boxed{2\pi i}$$

Example: Take the integral ~~of~~ $\oint_C \frac{dz}{(z-z_0)^n}$

when ~~[illegible]~~ C is any simple closed curve enclosing z_0.

Take a circle of radius r around z_0 that is inside C. Call this circle C_1. Its equation is $z = z_0 + re^{it}$ $\quad 0 \leq t \leq 2\pi$

Then $z - z_0 = re^{it}$ and $dz = rie^{it}$

So,

$$\oint_C \frac{dz}{(z-z_0)^n} = \oint_{C_1} \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{rie^{it}}{r^n e^{int}}\, dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{(1-n)it}\, dt$$

$$= \frac{i}{(1-n)ir^{n-1}} \left[e^{(1-n)it} \right]_0^{2\pi}$$

$$= \frac{1}{(1-n)r^{n-1}} [1 - 1] = 0 \quad \text{if } n \neq 1$$

if $n = 1$ then $\frac{i}{r^{n-1}} \int_0^{2\pi} e^{(1-n)it}\, dt = i \int_0^{2\pi} dt = 2\pi i$

So

$$\oint \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Example: Evaluate $\displaystyle\oint_C \frac{5z+7}{z^2+2z-3}\,dz$

where C is $|z-2|=2$.

$$z^2+2z-3=(z-1)(z+3)$$

The function is not analytic at $z=1$ and $z=-3$.

But only $z=1$ is inside C.

$$\frac{5z+7}{z^2+2z-3}=\frac{A}{z-1}+\frac{B}{z+3}$$

$$A=\left.\frac{5z+7}{z+3}\right|_{z=1}=3$$

$$B=\left.\frac{5z+7}{z-1}\right|_{z=-3}=2$$

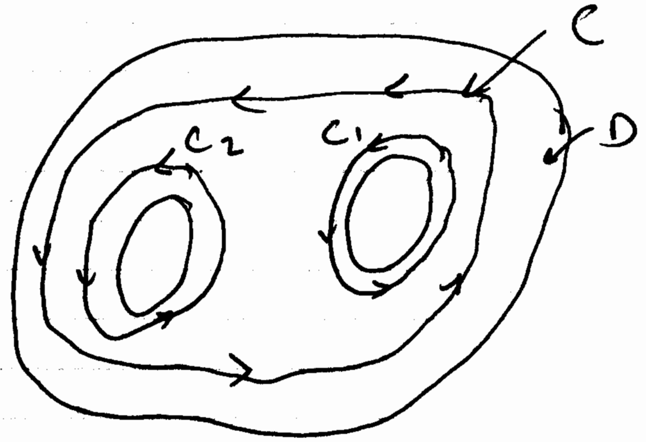
So

$$\frac{5z+7}{z^2+2z-3}=\frac{3}{z-1}+\frac{2}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz=3\oint_C\frac{dz}{z-1}+2\oint_C\frac{dz}{z+3}$$

$$=3(2\pi i)+2(0)=\boxed{6\pi i}$$

For a triply connected domain, i.e., a domain with two holes



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

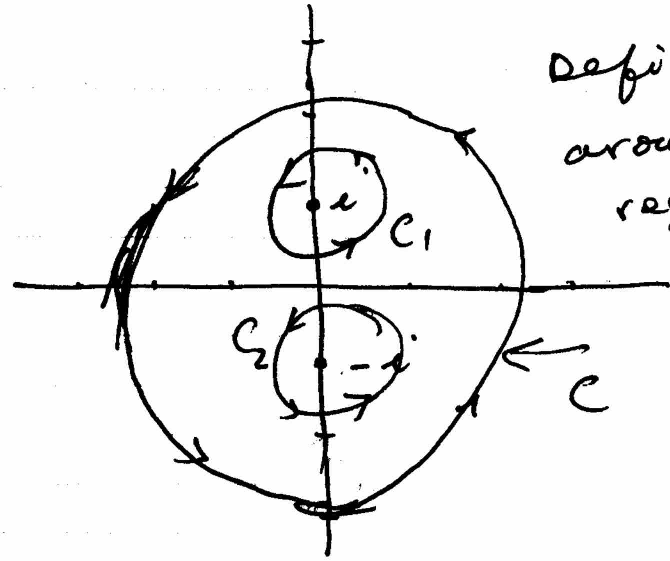
in general: (for a multiply connected domain):

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

Example: Evaluate $\oint_C \frac{dz}{z^2+1}$ over $|z|=3$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i}$$

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i} \oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$



Define C_1 and C_2 around i and $-i$, respectively.

Now

$$\oint_C \frac{dz}{z^2+1} = \oint_{C_1} \frac{dz}{z^2+1} + \oint_{C_2} \frac{dz}{z^2+1}$$

$$\oint_{C_1} \frac{dz}{z^2+1} = \frac{1}{2i} \cdot \oint_{C_1} \frac{1}{z-i}\, dz - \frac{1}{2i} \oint_{C_1} \frac{1}{z+i}\, dz$$

$$= \frac{2\pi i}{2i} - \frac{0}{2i} = \pi$$

$$\oint_{C_2} \frac{dz}{z^2+1} = \frac{1}{2i} \oint_{C_1} \frac{dz}{z-i} - \frac{1}{2i} \cdot \oint_{C_1} \frac{dz}{z+i} = \frac{0}{2i} - \frac{1}{2i} 2\pi i = -\pi$$

So

$$\boxed{\oint_C \frac{dz}{z^2+1} = \pi - \pi = 0}$$