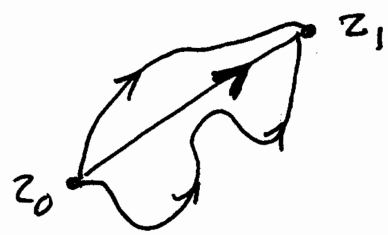
X Lecture 6, July 20, 2006

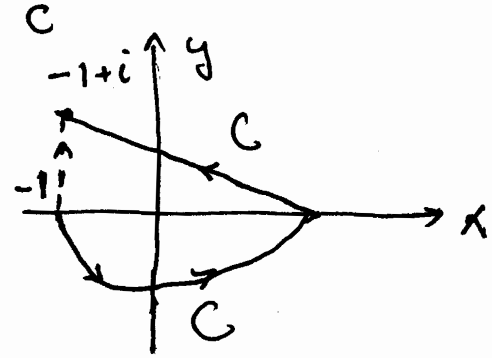
Independence of the Path

Let z_0 and z_1 be two points in a domain D. A contour integral $\int_C f(z)\,dz$ is said to be <u>independent of path</u> if its value is the same for any contour in D starting at z_0 and ending at z_1



<u>Theorem</u>: If f is analytic in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of the path C.

<u>Example</u>: $\int_C z\bar{z}\,dz$ where C is the contour shown as



instead of calculating the integral along C we can calculate it along C_1: a line connecting $z=-1$ to $z=-1+i$ (the dashed line)

$$\int_C 2z\,dz = \int_{C_1} 2z\,dz$$

The reason is that C and $-C_1$ form a closed contour an

$$\int_C 2z\,dz + \int_{-C_1} 2z\,dz = 0$$

and $$\int_C 2z\,dz = \int_{C_1} 2z\,dz$$

C_1 can be represented as $x=-1$ and $y=t$
$0 \leq t \leq 1$

or $z(t) = -1 + it$. So, $z' = i$ and,

$$\int_{C_1} 2z\,dz = \int_0^1 2(-1+it)i\,dt = -2i\int_0^1 dt - 2\int_0^1 t\,dt$$

$$= -2i - 2\frac{t^2}{2}\Big|_0^1 = \boxed{-1-2i}$$

Antiderivative

if f is Continuous in a domain D and there is a function F such that $F'(z) = f(z)$, we say that F is the antiderivative of f.

for example:

1) Take $F(z) = \cos z$ then $F'(z) = -\sin z$

So $\cos z$ is the antiderivative of $-\sin z$.

2) Take $F(z) = z^2 \Rightarrow F'(z) = 2z$.

So z^2 is antiderivative of $2z$.

Fundamental Theorem of antiderivatives

If f is continuous in a domain D and F is antiderivative of f. Then for any contour connecting points z_0 and z_1 (starting at z_0 and ending at z_1), we have:

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$

Let's now re-do the previous example:

$$\int_C 2z\,dz = z^2 \Big|_{-1}^{-1+i} = (-1+i)^2 - (-1)^2 = -1-2i$$

Example: Evaluate $\int_C \cos z\, dz$ where C is any contour with starting point $z=0$ and end point $z=2+i$

$$\int_C \cos z\, dz = \int_0^{2+i} \cos z\, dz = \sin z \Big|_0^{2+i} = \sin(2+i) - \sin(0)$$

$$= \sin(2+i)$$

$$= 1.4031 - 0.4891i$$

The last equality is the result of using,

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\sin(2+i) = \sin(2)\cosh(1) + i\cos(2)\sinh(1)$$

$$= 1.4031 - i\, 0.4891$$

The proof of $\sin z = \sin x \cosh(y) + i\cos(x)\sinh(y)$ is as follows:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix}\cdot e^{-y} - e^{-ix}\cdot e^{y}}{2i}$$

$$= \frac{(\cos x + i\sin x)e^{-y} - (\cos x - i\sin x)e^{y}}{2i}$$

$$= \sin x \frac{e^{y}+e^{-y}}{2} + i\cos x \frac{e^{y}-e^{-y}}{2}$$

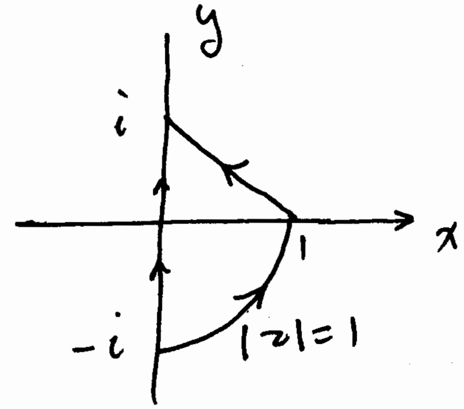
$$= \sin x \cosh(y) + i\cos x \sinh(x)$$

Existence of antiderivative

If f is analytic in a simply connected domain D, then f has an antiderivative F in D, i.e., there is a function F such that

$$F'(z) = f(z) \quad \text{for all } z \in D.$$

<u>Example</u>: Evaluate $\int_C (4z-1)\,dz$ where C is shown as



apart from difficult way of evaluating this integral by taking the integral over two sections of C, there are two other alternatives:

1) Finding the integral over an alternate simpler path (say the straight line connecting $-i$ to $+i$)
2) Using antiderivative.

1) Calculating along C_1 defined as

$x = 0 \quad , y = t \qquad -1 \le t \le 1$

$z(t) = it \qquad -1 \le t \le 1$

$z' = i$

$$\int_{-1}^{1} (4it - 1)\, i\, dt = -4\int_{-1}^{1} t\, dt - i\int_{-1}^{1} dt$$

$$= -2t^2\Big|_{-1}^{1} - 2i$$

$$= -2(1-1) - 2i = \boxed{-2i}$$

2) Using antiderivative:

$$\int_C (4z-1)\, dz = (2z^2 - z)\Big|_{-i}^{i} = (-2-i) - (-2+i)$$

$$= -2 - i + 2 - i = \boxed{-2i}$$

Example: Evaluate $\displaystyle\int_{\pi}^{\pi+2i} \sin\left(\frac{z}{2}\right) dz$

$$\int_{\pi}^{\pi+2i} \sin\left(\frac{z}{2}\right) dz = -2\cos\left(\frac{z}{2}\right)\Big|_{\pi}^{\pi+2i} = -2\cos\left(\frac{\pi}{2}+i\right) + \cos\left(\frac{\pi}{2}\right)$$

$$= -2\cos\left(\frac{\pi}{2}+i\right)$$

$$= -2\left[\cos\frac{\pi}{2}\cosh(1) - i\sin\frac{\pi}{2}\sinh(1)\right]$$

$$= 2i\sinh(1) = \boxed{2.35i}$$

Example: Evaluate $\displaystyle\int_{\pi i}^{2\pi i} \cosh(z)\,dz$

$$\int_{\pi i}^{2\pi i} \cosh(z)\,dz = \sinh(z)\Big|_{\pi i}^{2\pi i} = \sinh(2\pi i) - \sinh(\pi i)$$

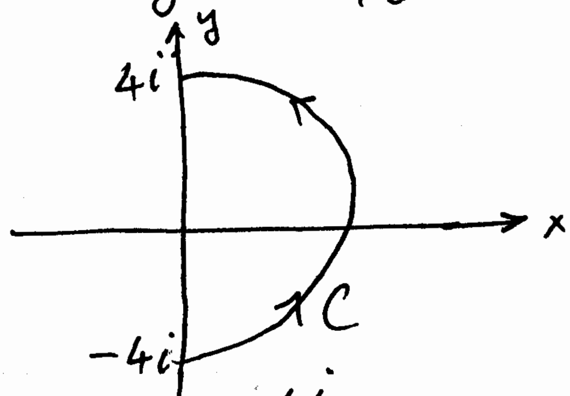
$$\sinh(z) = \frac{e^{z} - e^{-z}}{2i}$$

$$\sinh(2\pi i) = \frac{e^{2\pi i} - e^{-2\pi i}}{2i} = \frac{1-1}{2i} = 0$$

$$\sinh(\pi i) = \frac{e^{\pi i} - e^{-\pi i}}{2i} = \frac{-1-(-1)}{2i} = 0$$

So: $\boxed{\displaystyle\int_{\pi i}^{2\pi i} \cosh(z)\,dz = 0}$

Example: Evaluate $\displaystyle\int_C \frac{1}{z}\,dz$ where C is defined by $z = 4e^{it}$ $\quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$



$$\int_C \frac{1}{z}\,dz = \int_{-4i}^{4i} \frac{1}{z}\,dz = Ln(z)\Big|_{-4i}^{4i} = Ln(4i) - Ln(-4i)$$

But $4i = 4e^{i\pi/2}$ and $-4i = 4e^{-i\frac{\pi}{2}}$

So,

$$\int_C \frac{1}{z}\,dz = \mathrm{Ln}(4e^{i\pi/2}) - \mathrm{Ln}(4e^{-i\pi/2})$$

$$= \mathrm{Ln}(4) + i\frac{\pi}{2} - \left(\mathrm{Ln}(4) - i\frac{\pi}{2}\right)$$

$$= \boxed{i\pi}$$

Cauchy's integral formula

~~It~~ If f is analytic in a simply connected domain D and C is a simple closed contour entirely inside D and z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz$$

Proof by principle of deformation of contours, we can evaluate the integral over any contour C_1 enclosing z_0 and fully inside C. We choose a circle of small radius centered at z_0 as C_1.

$$\oint \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z_0) - f(z_0) + f(z)}{z - z_0}\,dz$$

$$= f(z_0) \int_{C_1} \frac{dz}{z - z_0} + \int_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

We have shown before that

$$\int_{C_1} \frac{dz}{z-z_0} = 2\pi i$$

So,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

Since $f(z)$ is continuous at z_0 for any $\varepsilon > 0$ there exists a δ such that

$$|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$$

Now if we choose C_1 as the circle with radius $\frac{\delta}{2}$, i.e. $|z-z_0| \doteq \frac{\delta}{2} < \delta$, we have

$$\left| \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \leq \frac{\varepsilon}{\delta/2}\, 2\pi\left(\frac{\delta}{2}\right) = 2\pi\varepsilon$$

where we have used the Theorem 18.3 page 830 which says if the absolute value of a function $f(z)$ is less than M, then the absolute value of its integral over a path of length L is less than ML.

By making $\varepsilon > 0$ arbitrarily small, we can make the integral $\oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \to 0$

So,

$$\int \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Example: Evaluate $\displaystyle\oint_C \frac{z^2-4z+4}{z+i}\,dz$

where C is $|z| = 2$.

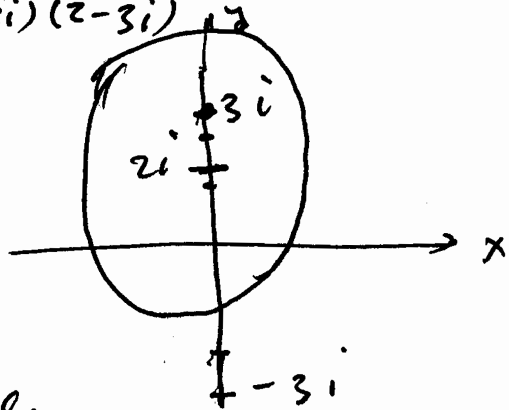
Let $f(z) = z^2-4z+4$, $z_0 = -i$ is inside C

$$\oint_C \frac{z^2-4z+4}{z+i}\,dz = 2\pi i\, f(-i) = 2\pi i\,(3+4i)$$
$$= \boxed{2\pi(-4+3i)}$$

Example: Evaluate $\displaystyle\oint_C \frac{z}{z^2+9}\,dz$

where C is $|z-2i| = 4$

$$\oint_C \frac{z}{z^2+9}\,dz = \oint_C \frac{z}{(z+3i)(z-3i)}\,dz$$



Since $3i$ is inside the circle and $-3i$ is outside, we write the integral as:

$$\oint_C \frac{\frac{z}{z+3i}}{z-3i}\,dz \quad \text{, i.e., we choose}$$

$$f(z) = \frac{z}{z+3i} \quad \text{and} \quad z_0 = 3i$$

Then

$$\oint_C \frac{z}{z^2+9}\,dz = \oint_C \frac{\frac{z}{z+3i}}{z-3i}\,dz = 2\pi i\, f(3i)$$

$$= 2\pi i\left(\frac{3i}{6i}\right) = \boxed{\pi i}$$

Cauchy integral for Derivatives

Let f be analytic in a simply connected domain D and C be a simple closed contour entirely in D. If z_0 is any point inside C then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Example: Evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$ where C is $|z|=1$.

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \oint_C \frac{z+1}{z^3(z+4)}\,dz$$

the integrand $\frac{z+1}{z^4+4z^3}$ is not ~~analytic at~~ analytic at $z=-4$ and $z=0$. But only $z=0$ is inside C. So, we choose

$$f(z) = \frac{z+1}{z+4}, \quad z_0 = 0, \quad n = 2.$$

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \oint_C \frac{\frac{z+1}{z+4}}{z^3}\,dz = \frac{2\pi i}{2!} f''(0)$$

$$f'(z) = \frac{z+4-(z+1)}{(z+4)^2} = \frac{3}{(z+4)^2}$$

$$f''(z) = \frac{-2(z+4)\times 6}{(z+4)^4} = \frac{-6}{(z+4)^3}$$

$$f''(0) = \frac{-6}{64} = \frac{-3}{32}$$

So

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2}\left(-\frac{3}{32}\right) = \boxed{\frac{-3\pi}{32}i}$$