Lecture 7, July 25, 2006

Some review exercises:

1) Evaluate

$$\oint_C \frac{2z+5}{z^2-2z}\,dz$$

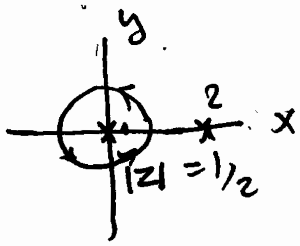
for C:

a) $|z| = \frac{1}{2}$ b) $C: |z+1| = 2$

c) $|z-3| = 2$ d) $|z+2i| = 1$

The integrand has two discontinuities $z_0 = 0$ and $z = 2$

a)

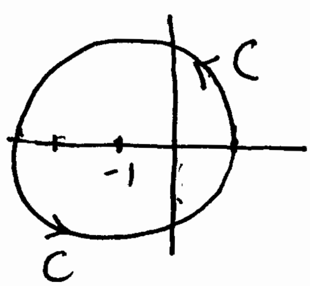


only $z = 0$ is inside C. So,

$$\oint_C \frac{2z+5}{z^2-2z}\,dz = \oint_C \frac{\frac{2z+5}{z-2}}{z}\,dz = 2\pi i f(0) = 2\pi i \cdot \frac{-5}{2}$$

$$= -5\pi i$$

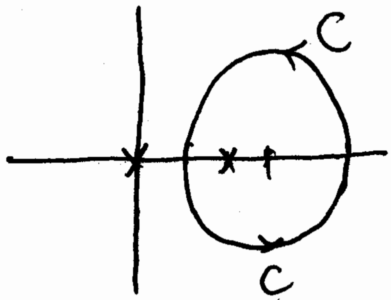
b)



since again only $z = 0$ is inside the contour, we get

$$\oint \frac{2z+5}{z^2-2z}\,dz = -5i$$

c)

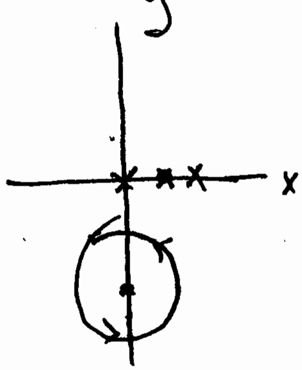


here, only $z=2$ is inside C, so

$$\oint_C \frac{z+5}{z^2-2z}\,dz = \oint_C \frac{\frac{z+5}{z}}{z-2}\,dz = 2\pi i\left(\frac{z+5}{z}\right)\Big|_{z=2}$$

$$= 9\pi i$$

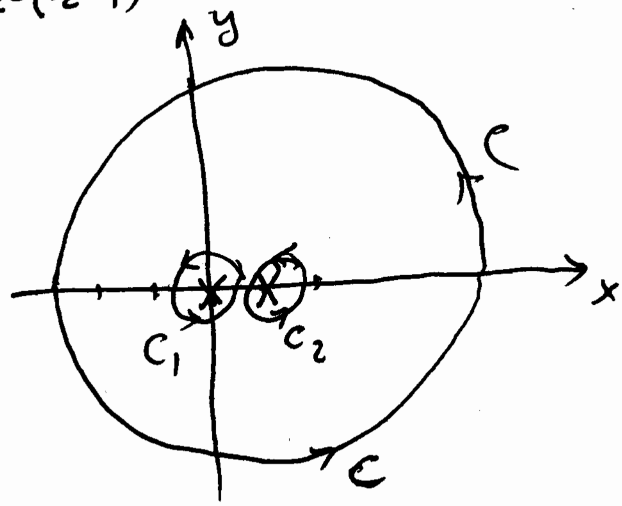
d)



Since neither $z=0$ nor $z=2$ is inside C

$$\oint_C \frac{z+5}{z^2-2z}\,dz = 0$$

<u>Example</u>: Evaluate $\oint_C \frac{dz}{z^3(z-1)^2}\,dz$ where C is $|z-2|=5$.



The integrand is discontinuous at $z=0$ and $z=1$. Both of these points are inside C.

To be able to evaluate the integral, we define two contours C_1 and C_2 each containing one of the points $z=0$ and $z=1$ and both fully contained in C. Then, using Cauchy-Goursat Theorem,

$$\oint_C \frac{dz}{z^3(z-1)^2} = \oint_{C_1} \frac{dz}{z^3(z-1)^2} + \oint_{C_2} \frac{dz}{z^3(z-1)^2}$$

Now using Cauchy's integral formula,

$$\oint_{C_1} \frac{dz}{z^3(z-1)^2} = \frac{\frac{1}{(z-1)^2}}{z^3}\,dz = \frac{2\pi i}{2!} f''(z)\Big|_{z=0}$$

where $f(z) = \frac{1}{(z-1)^2} \Rightarrow f''(z) = \frac{6}{(z-1)^4}$

So,

$$\oint_{C_1} \frac{dz}{z^3(z-1)^2} = \frac{2\pi i}{2}\,\frac{6}{1} = 6\pi i$$

We also have

$$\oint_{C_2} \frac{dz}{z^3(z-1)^2} = \oint_{C_2} \frac{1/z^3}{(z-1)^2}\,dz = 2\pi i\, f'(z)\Big|_{z=1}$$

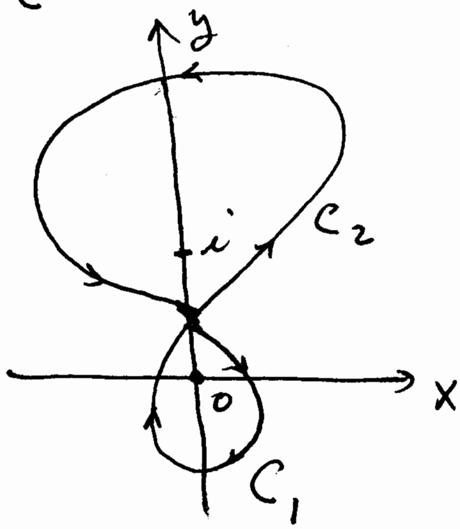
where $f(z) = \frac{1}{z^3} \Rightarrow f'(z) = -\frac{3}{z^4}$

$$\oint_{C_2} \frac{dz}{z^3(z-1)^2} = 2\pi i\,(-3) = -6\pi i$$

So $\oint_C \frac{dz}{z^3(z-1)^2} = \boxed{6\pi i - 6\pi i = 0}$

Example: Evaluate $\oint_C \frac{z^2+3}{z(z-i)^2}\,dz$ where

C is shown as



C is not simply closed.

But we can consider it

as the union of

C_1 an C_2 where

C_1 encircles $z=0$ and C_2 encircles $z=i$.

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\oint_{C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\,dz$$

$$= -2\pi i\left[\frac{z^3+3}{(z-i)^2}\right]\Bigg|_{z=0} + 2\pi i\left[\frac{d}{dz}\left(\frac{z^3+3}{z}\right)\right]\Bigg|_{z=i}$$

$$= -2\pi i(-3) + 2\pi i(2i+3) = \boxed{4\pi(-1+3i)}$$

Chapter 19: Series and Residues

<u>Sequences</u>: A sequence is a function whose domain is the set of positive integers. It is something similar to arrays in computer programs. Take for example:

$$z_n = 1 + i^n$$

$$z_1 = 1 + i, \quad z_2 = 0, \quad z_3 = 1 - i, \quad z_4 = 2, \ \cdots$$

We say a sequence is <u>convergent</u> if

$$\lim_{n \to \infty} z_n = L.$$

This means that given any number $\varepsilon > 0$ (no matter how small it may be) we should be able to find an integer N such that

$$|z_n - L| < \varepsilon \quad \text{for } n > N.$$

<u>Example</u>: The sequence $z_n = 1 + i^n$ is divergent since it does not approach a fixed point.

Theorem: A sequence $\{z_n\}$ converges to a complex number L if and only $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(L)$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(L)$.

Example: Consider $z_n = \dfrac{ni}{n+2i}$.

$$z_n = \frac{ni}{n+2i} = \frac{ni(n-2i)}{(n+2i)(n-2i)} = \frac{2n}{n^2+4} + i\,\frac{n^2}{n^2+4}$$

as $n \to \infty$

$$\mathrm{Re}(z_n) = \frac{2n}{n^2+4} \to 0$$

and

$$\mathrm{Im}(z_n) = \frac{n^2}{n^2+4} \to 1$$

So $z_n \to L = i$

An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_k + \cdots$$

is convergent if the sequence of partial sums of its terms $S_n = z_1 + z_2 + \cdots + z_n$ is convergent. If $S_n \to L$ as $n \to \infty$, we say that sum of the series is L.

Geometric Series

$$\sum_{k=1}^{\infty} a z^{k-1} = a + az + az^2 + \cdots + az^{k-1} + \cdots$$

We have

$$S_n = a + az + az^2 + \cdots + az^{n-1}$$

multiply S_n by z to get,

$$z S_n = az + az^2 + az^3 + \cdots + az^n$$

$$= -a + \underbrace{a + az + az^2 + \cdots + az^{n-1}}_{S_n} + az^n$$

So,

$$z S_n = S_n + az^n - a$$

or

$$S_n = \frac{a(1-z^n)}{1-z}$$

If $|z| < 1$ then as $n \to \infty$, $z^n \to 0$ and the sum Converges to $\dfrac{a}{1-z}$.

When $|z| \geq 1$ the series diverges.

Two special Cases of geometric Series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

and

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

for $|z| < 1$ are two important and useful series in the rest of the course.

Also, with $a = 1$ the sum S_n will be

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$$

or equivalently, we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1-z}$$

Example: The series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series with

$$a = \frac{1+2i}{5} \quad \text{and} \quad z = \frac{1+2i}{5}$$

$|z| = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} < 1$. So, the series converges.

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{a}{1-z} = \frac{\frac{(1+2i)}{5}}{1 - \frac{1+2i}{5}} = \boxed{\frac{i}{2}}$$

Some theorems for testing the convergence of series:

Theorem: If $\sum_{k=1}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n \to 0$.

Theorem: If $\lim_{n\to\infty} z_n \neq 0$ then $\sum_{k=1}^{\infty} z_n$ diverges.

Definition: (Absolute Convergence):
An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be <u>absolutely convergent</u> if $\sum_{k=1}^{\infty} |z_k|$ converges.

Theorem: Absolute convergence implies convergence.

<u>Example</u>: The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent since

$$\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}$$

<u>Ratio Test</u>

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms and

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L.$$

if $L < 1$ then series converges absolutely
if $L > 1$ or $L = \infty$, then the series is divergent
if $L = 1$, the test is inconclusive.

Root Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms

such that

$$\lim_{n \to \infty} \sqrt[n]{|z_n|} = L$$

if $L < 1$ then the series converges absolutely.
if $L > 1$ or $L = \infty$, the series diverges
if $L = 1$, the test is inconclusive.

Example: Is $\sum_{k=1}^{\infty} (1-i)^{k-1}$ convergent or divergent

$$\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n \to \infty} \left| \frac{(1-i)^n}{(1-i)^{n-1}} \right| = |1-i| = \sqrt{2} > 1$$

So, the series is divergent.

Example: is the series $\sum_{k=1}^{\infty} 4i\left(\frac{1}{3}\right)^{k-1}$ convergent or divergent?

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{4i\left(\frac{1}{3}\right)^n}{4i(1/3)^{n-1}}\right| = \frac{1}{3} < 1$$

So, the series is convergent.

We can find the sum as

$$\sum_{k=1}^{\infty} 4i\left(\frac{1}{3}\right)^{k-1} = \frac{4i}{1-1/3} = \frac{4i}{\frac{2}{3}} = \boxed{6i}$$

Example: Is the series $\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}}$ convergen? if yes find its sum.

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{i^{n+1}}{(1+i)^n}}{\frac{i^n}{(1+i)^{n-1}}}\right| = \left|\frac{i}{1+i}\right| = \frac{1}{\sqrt{2}} < 1$$

So, it is convergent

$$\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}} = \frac{i^2}{1+i} + \frac{i^2}{i+1}\cdot\frac{i}{1+i} + \frac{i^2}{i+1}\left(\frac{i}{1+i}\right)^2 + \cdots$$

So, this is a geometric series with

$$a = \frac{i^2}{1+i} = \frac{-1}{1+i} \quad \text{and} \quad z = \frac{i}{1+i}$$

So $$\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}} = \frac{a}{1-z} = \frac{-1/1+i}{1-\frac{i}{1+i}} = \frac{-1}{1+i-i} = -1$$

Power Series:

Any series of the form

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

is called a power series centered at z_0.

Circle of Convergence

Every complex power series has a radius of convergence R. When $0 < R < \infty$, a complex power series has a circle of convergence defined as $|z - z_0| = R$. The power series is absolutely convergent for all z satisfying $|z - z_0| < R$ and diverges for $|z - z_0| > R$.

Example: Consider power series

$$\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}.$$

Using ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \left| \frac{nz}{n+1} \right| = |z|$$

The series converges absolutely for $|z|<1$. The circle of convergence is $|z|=1$. The radius of convergences is $R=1$.

Consider the power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z-z_0|$$

So if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$ then $R=\frac{1}{L}$

if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then $R\to\infty$.

if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $R=0$.

Example: Find the radius of convergence for

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1-i)^k}{k!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

So, the radius of convergence is ∞. This means that this power series converges everywhere.

<u>Example</u>: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$

$$a_n = \left(\frac{6n+1}{2n+5}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{6n+1}{2n+5} = 3$$

So the radius of convergence is $R = \frac{1}{3}$ and the circle of convergence is $|z-2i| = \frac{1}{3}$.