Lecture 8, July 27, 2006

Taylor Series

Before discussing Taylor series for Complex functions, we present a few theorems about power series.

Theorem: A power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ represents a continuous function f within its region of convergence, $|z-z_0|=R$, $R \neq 0$.

Theorem: A power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ can be integrated in its circle of convergence $|z-z_0|=R$, $R\neq 0$ for every contour entirely within its circle of convergence.

Theorem: A power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ can be differentiated term by term within its circle of convergence $|z-z_0|=R$, $R \neq 0$

Taylor Series

Assume $f(z)$ is the function that a power series represent ~~for~~ in $|z-z_0|<R$, $R\neq 0$. Then

$$f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Let $z = z_0$ to get

$f(z_0) = a_0$ since all other terms have $(z-z_0)$ so they become zero.

Then take derivative of $f(z)$

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$$

letting $z = z_0$, we get

$$f'(0) = a_1$$

Taking Second derivative,

$$f''(z) = 2a_2 + 3\times 2\, a_3(z-z_0) + \cdots$$

and for $z = z_0$,

$$f''(0) = 2a_2 = 2!\, a_2$$

Similarly

$$f'''(0) = 3\times 2\, a_3 = 3!\, a_3$$

or in general

$$f^{(n)}(z_0) = n!\, a_n \Rightarrow a_n = \frac{f^{(n)}(z_0)}{n!}$$

So, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

This is called Taylor Series Centered at z_0.

If $z_0 = 0$, i.e., for Taylor series centered at 0,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

This is called a <u>Maclaurin series</u>

We have shown that a power series with non zero radius of convergence represents an analytic function.

Conversely one can show that if f (an arbitrary function) is analytic in a domain D and z_0 be a point in D. Then f has a series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k$$

valid for the largest C with center at z_0 and radius R that lies entirely within D.

Example $f(z) = e^z$

$$f'(z) = f''(z) = f'''(z) = \cdots = f^{(k)}(z) = \cdots = e^z$$

and $f^{(k)}(0) = 1$ all k

so $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

Example: $\sin z$

$$f'(z) = \cos z\ ,\quad f''(z) = -\sin z\ ,\quad f'''(z) = -\cos z, \ldots$$

So

$$a_0 = f(0) = 0$$

$$a_1 = f'(0) = 1$$

$$a_2 = f''(0) = 0$$

$$a_3 = \frac{f'''(0)}{3!} = -\frac{1}{3!} \qquad \text{etc.}$$

So

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

Similarly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

Example: Find the Maclaurin series expansion of

$$f(z) = \frac{1}{(1-z)^2}$$

We had

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

So

$$\frac{1}{(1-z)^2} = (1 + z + z^2 + \cdots)(1 + z + z^2 + \cdots) = 1 + 2z + 3z^2 + \cdots$$

$$= \sum_{k=0}^{\infty} k z^{k-1}$$

The radius of convergence is $R = 1$.

Example: Find Taylor series expansion for

$f(z) = \dfrac{1}{1-z}$ around $z_0 = 2i$

$$f'(z) = \frac{1}{(1-z)^2}, \quad f''(z) = \frac{2\times 1}{(1-z)^3}, \quad f'''(z) = \frac{3\times 2}{(1-z)^4}$$

in general $f^{(n)}(z) = \dfrac{n!}{(1-z)^{n+1}}$

so,

$$f^{(n)}(2i) = \frac{n!}{(1-2i)^{n+1}} \quad \text{and} \quad a_n = \frac{f^{(n)}(2i)}{n!} = \frac{1}{(1-2i)^{n+1}}$$

so,

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{1}{(1-2i)^{k+1}} (z-2i)^k$$

The radius of convergence of this power series expansion is the distance between $z_0 = 2i$ and the nearest singularity of $f(z)$, i.e., $z = 1$ which is $R = \sqrt{2^2+1} = \sqrt{5}$

Example: Expand $f(z) = \dfrac{z}{1+z}$ using Maclaurin series

$$f(z) = z\frac{1}{1+z} = z\left[1 - z + z^2 - z^3 + \cdots\right]$$

$$= z - z^2 + z^3 - z^4 + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} z^k$$

$R = 1$ both by observation and since the distance between $z_0 = 0$ and $z = -1$ is 1.

Examples on MacLaurin series:

Example: Find the Maclaurin series for

$$f(z) = \frac{1}{(1+2z)^2}$$

direct way:

$$f(z) = \frac{1}{(1+2z)^2} \qquad f(0) = 1$$

$$f'(z) = \frac{-2 \times 2}{(1+2z)^3} \qquad f'(0) = -2$$

$$f''(z) = \frac{+2 \times 3 \times 2^2}{(1+2z)^4} \qquad f''(0) = 3!$$

$$\vdots$$

$$f^{(n)}(z) = \frac{(n+1)!}{(1+2z)^{2n}}(-2)^n \qquad f^{(n)}(0) = (n+1)!(-2)^n$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} (-2)^k (k+1) z^k$$

$$= \sum_{k=0}^{\infty} (-1)^k (k+1) (2z)^k$$

tricky way:

we now that $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$

so

$$\frac{1}{1+2z} = 1 - (2z) + (2z)^2 - (2z)^3 + \cdots$$

and

$$\frac{1}{(1+2z)^2} = \left[1-(2z)+(2z)^2-(2z)^3+\cdots\right]^2$$

$$=\left[1-(2z)+(2z)^2-\cdots\right]\left[1-2z+(2z)^2-\cdots\right]$$

$$=1-2(2z)+3(2z)^2-4(2z)^3+5(2z)^4-\cdots$$

$$=\sum_{k=0}^{\infty}(-1)^k(k+1)(2z)^k$$

$R = 1/2$, the distance between zero and $z=-\frac{1}{2}$

Example: $f(z) = e^{-2z}$

We now that

$$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

Substitute $-2z$ for z:

$$f(z) = 1-(2z)+\frac{(2z)^2}{2!}-\frac{(2z)^3}{3!}+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(2z)^k$$

$R = \infty$ (why?)

We could also have done it directly

$$f(z) = e^{-2z} \qquad f(0) = 1$$

$$f'(z) = -2e^{-2z} \qquad f'(0) = -2$$

$$f''(z) = 4e^{-2z} \qquad f''(0) = 4$$

$$f'''(z) = -8e^{-2z} \qquad f'''(0) = -8$$

$$\vdots$$

$$f^{(n)}(z) = (-2)^n e^{-2z} \qquad f^{(n)}(0) = (-2)^n$$

So,

$$f(z) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} z^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (2z)^k$$

<u>Example</u>: Find the Maclaurin series expansion for $f(z) = \sinh(z)$

$$f(z) = \sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$$

$$e^{z} - e^{-z} = 2z + 2\frac{z^3}{3!} + 2\frac{z^5}{5!} + \cdots$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

$R = \infty$ since $\sinh(z)$ is analytic everywhere.

Some examples of Taylor series expansion:

<u>Example</u>: Find Taylor series expansion of $f(z) = \frac{1}{z}$ centered at $z_0 = 1+i$

$$f(z) = \frac{1}{z} \qquad f(z_0) = \frac{1}{1+i}$$

$$f'(z) = \frac{-1}{z^2} \qquad f'(z_0) = \frac{-1}{(1+i)^2}$$

$$f''(z) = \frac{2}{z^3} \qquad f''(z_0) = \frac{2}{(1+i)^3}$$

$$f'''(z) = \frac{-3\times 2}{z^4}$$

$$\vdots$$

$$f^{(n)}(z) = (-1)^n \frac{n!}{z^{n+1}} \qquad f^{(n)}(z_0) = (-1)^n \frac{n!}{(1+i)^{n+1}}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(1+i)^{k+1}} (z-1-i)^k$$

$R = \sqrt{2}$ (why?)

Example: Find the Taylor Series expansion for $f(z) = \frac{1}{1+z}$ at $z_0 = -i$

$$f(z) = \frac{1}{1+z} \qquad f(-i) = \frac{1}{1-i}$$

$$f'(z) = \frac{-1}{(1+z)^2} \qquad f'(-i) = \frac{-1}{(1-i)^2}$$

$$f''(z) = \frac{+2}{(1+z)^3} \qquad f''(-i) = \frac{2}{(1-i)^3}$$

$$f'''(z) = \frac{-3!}{(1+z)^4} \qquad f'''(-i) = \frac{-3!}{(1-2i)^4}$$

in general $f^{(n)}(-i) = (-1)^n \frac{n!}{(1-2i)^{n+1}}$

so,

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(1-2i)^{k+1}} (z-i)^k$$

$R = \sqrt{2}$ distance from $z_0 = -i$ to discontinuity of $f(z)$ at -1.

Example: Find the Taylor Series expansion of $f(z) = \dfrac{z-1}{3-z}$ at $z_0 = 1$

$f(z) = \dfrac{z-1}{3-z}$ $\qquad f(1) = 0$

$f'(z) = \dfrac{2}{(3-z)^2}$ $\qquad f'(1) = \dfrac{1}{2}$

$f''(z) = \dfrac{+4}{(3-z)^3}$ $\qquad f''(1) = \dfrac{1}{2}$

$f'''(z) = \dfrac{12}{(3-z)^4}$ $\qquad f'''(\phi) = \dfrac{3}{4}$

$\vdots$

$f^{(n)}(z) = \dfrac{2n!}{(3-z)^{n+1}}$ $\qquad f^{(n)}(1) = \dfrac{2n!}{2^{n+1}} = \dfrac{n!}{2^n}$

So:

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{2^k}(z-1)^k$$

R is the distance between $z_0 = 1$ and 3

$R = 2$

Example: Find the Taylor series expansion of $f(z) = \cos z$ at $z_0 = \frac{\pi}{4}$

$f(z) = \cos z \qquad f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(z) = -\sin z \qquad f'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f''(z) = -\cos z \qquad f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f'''(z) = \sin z \qquad f'''(\pi/4) = \frac{\sqrt{2}}{2}$

$f^{(n)}(\frac{\pi}{4}) = \pm\frac{\sqrt{2}}{2}$ where $+$ for $k = 0, 3, 4, \cancel{7}, 8, \ldots$

and $-$ for $k = 1, 2, 5, 6, \ldots$

So:

$$f(z) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(z - \frac{\pi}{4}) - \frac{\sqrt{2}}{2\times 2!}(z - \frac{\pi}{4})^2 + \frac{\sqrt{2}}{2\times 3!}(z - \frac{\pi}{4})^3 + \cdots$$

$R = \infty$ (why?)