Lecture 9, August 1, 2006

Singularities, Laurent Series

If $z_0$ is a singularity of a function $f(z)$, it is obvious that we cannot expand $f(z)$ using Taylor Series at $z=z_0$ since $f(z_0)=\infty$ and $f(z_0)$ is the first term in the expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} z^n .$$

However, if $z_0$ is an isolated singularity, i.e., $f(z)$ is only discontinuous at this point and analytic in any neighborhood of $z_0$, then we can find a series expansion for $f(z)$. This is called the Laurent Series.

Say $f(z)$ is written as $\frac{g(z)}{z-z_0}$, i.e., it has a singularity at $z=z_0$. Assuming $g(z)$ is analytic at $z_0$, we can expand $g(z)$ as

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k$$

Now,

$$f(z) = \frac{1}{z-z_0} \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k$$

or

$$f(z) = \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

in general we may have many negative power terms, i.e.,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

$$= \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

To clarify the idea consider the following example:

$$f(z) = \frac{\sin z}{z^3}$$

This function is not analytic at $z=0$.

However, we have a Taylor series expansion for $\sin z$,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

So

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

This series converges at all $z$ except $z=0$

i.e., it converges for $|z|>0$.

In general,

$$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

Converges in a ring defined by $r<|z-z_0|<R$.

Theorem (Laurent Series):

Let $f$ be analytic within an annular domain $D$ defined by $r<|z-z_0|<R$.

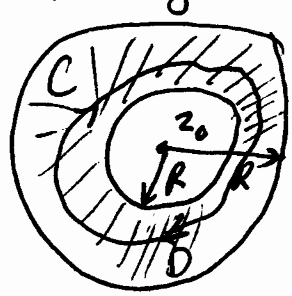
Then $f$ has a Laurent series expansion

$$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

valid for $r<|z-z_0|<R$ where

$$a_k=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz \qquad k=0,\pm 1,\pm 2,\cdots$$

where $C$ is a simple closed contour that lies entirely in $D$ and has $z_0$ inside.

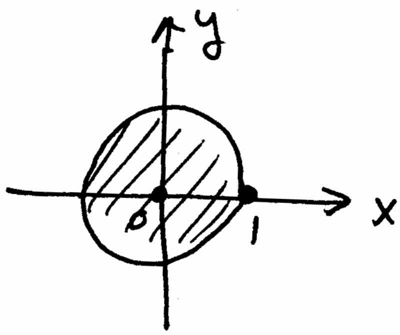


<u>Example</u>: Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series for

a) $0 < |z| < 1$ b) $|z| > 1$

c) $0 < |z-1| < 1$ d) $|z-1| > 1$

a)




$0 < |z| < 1$

here we want to expand in terms of powers of $z$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z}\,\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-z}$$

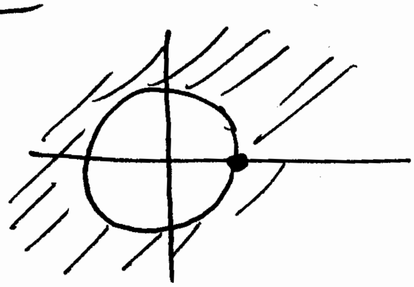
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$f(z) = -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= -\frac{1}{z} - 1 - z - z^2 - \cdots$$

This series converges for $0 < |z| < 1$.

b)



$|z| > 1$

So, we expand the series by finding a part that is a function of $\frac{1}{z}$ & converges ~~as~~ for $\left|\frac{1}{z}\right| < 1$

(equivalently $|z| > 1$)

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z}\frac{1}{z-1}=\frac{1}{z^2}\frac{1}{1-\frac{1}{z}}$$

$$\frac{1}{1-\frac{1}{z}}=1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+ \cdots$$

$$f(z)=\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots$$

c) $\quad 0<|z-1|<1$

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z-1}\frac{1}{z}$$

Now, we expand $\frac{1}{z}$ at $z_0=1$

$$\frac{1}{z}=\frac{1}{1+(z-1)}=1-(z-1)+(z-1)^2-(z-1)^3$$

So,

$$f(z)=\frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots$$

d) $\quad |z-1|>1 \quad$ again, we form a series in terms of $\frac{1}{z-1}$

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z-1}\frac{1}{z}=\frac{1}{z-1}\frac{1}{1+z-1}=\frac{1}{(z-1)^2}\frac{1}{1+\frac{1}{z-1}}$$

$$\frac{1}{1+\frac{1}{z-1}}=1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots$$

$$f(z) = \frac{1}{(z-1)^2}\left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \cdots\right]$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \cdots$$

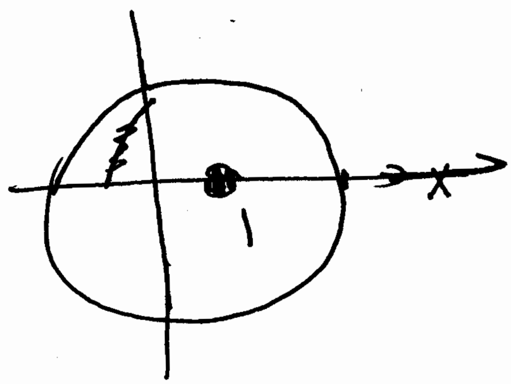
<u>Example</u>: Expand $f(z) = \dfrac{1}{(z-1)^2(z-3)}$

using Laurent series for

a) $0 < |z-1| < 2$

b) $0 < |z-3| < 2$

a)




$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\,\frac{1}{z-3}$$

we now expand $\dfrac{1}{z-3}$ at $z_0 = 1$

$$\frac{1}{z-3} = \frac{1}{-2+(z-1)} = \frac{-1}{2}\,\frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{4} + \cdots\right]$$

$$f(z) = -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

b) $0 < |z-3| < 2$

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3}\,\frac{1}{(z-1)^2}$$

we expand $\frac{1}{(z-1)^2}$ at $z_0 = 3$

$$g(z) = \frac{1}{(z-1)^2} \qquad g(3) = \frac{1}{2^2}$$

$$g'(z) = \frac{-2}{(z-1)^3} \qquad g'(3) = \frac{-2!}{2^3}$$

$$g''(z) = \frac{+3\times 2}{(z-1)^4} \qquad g''(3) = \frac{3!}{2^4}$$

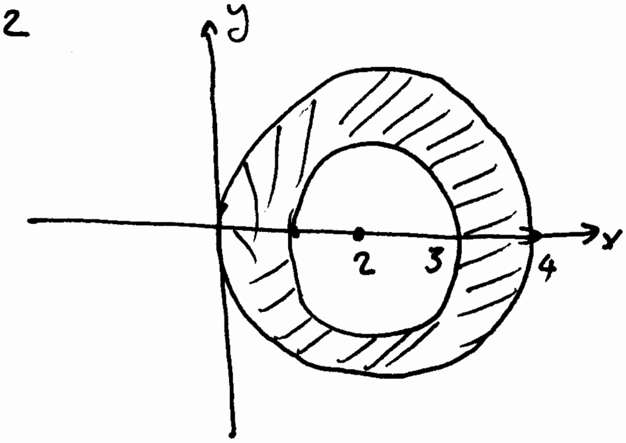
$$g^{(n)}(3) = \frac{(n+1)!}{2^{n+2}}$$

$$\frac{1}{(z-1)^2} = \frac{1}{4} - \frac{1}{4}(z-3) + \frac{3}{16}(z-3)^2 - \frac{1}{8}(z-3)^3 + \cdots$$

$$f(z) = \frac{1}{z-3}\left[\frac{1}{4} - \frac{1}{4}(z-3) + \frac{3}{16}(z-3)^2 - \frac{1}{8}(z-3)^3 + \cdots\right]$$

$$= \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \cdots$$

Ex. Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent Series valid for $1 < |z-2| < 2$




Let

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$-\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\,\frac{1}{1+\frac{z-2}{2}}$$

$$= -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right]$$

$$= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$

This series converges for $\left|\frac{z-2}{2}\right| < 1$ or $|z-2| < 2$

for $\dfrac{1}{z-1}$, we have

$$\frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\,\frac{1}{1+\frac{1}{z-2}}$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \cdots\right]$$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \cdots$$

This series Converges for $\left|\frac{1}{z-2}\right| < 1 \Rightarrow |z-2| > 1$

So $f(z) = -\frac{1}{z} + \frac{1}{z-1}$ converges ~~for~~ in the intersection of the two areas of convergence, i.e.,

$$1 < |z-2| < 2 .$$

$$f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \cdots$$

Example: Expand $f(z) = e^{3/z}$ in a Laurent series for $|z| > 0$.

We have

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

So,

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$$

which is valid for $|z| > 0$

**Zeros and Poles**

Assume that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

is the series expansion of $f(z)$ for $r < |z-z_0| < R$.

We can write $f(z)$ as,

$$f(z) = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

The part $\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$ is called the

Principal Part of $f(z)$ at $z = z_0$.

Isolated singularities can be classified as:

1) if the principal part is zero, i.e., all $a_{-k}$ are zero, then $z=z_0$ is called a removable singularity.

2) if the principal part has a finite number of nonzero terms, then $z=z_0$ is called a pole. If $a_{-n}$ is the last non-zero term, then $z=z_0$ is a pole of order $n$.

3) if the principal part has an infinite number of terms, then it is called an essential singularity.

So,

1) if $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ we say $z=z_0$ is a removable singularity.

2) if $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$ we say $z_0$ is a pole of order $n$.

3) if $n=1$, we have

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

and $z=z_0$ is called a simple pole.

4) if

$$\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

we say that $z=z_0$ is an <u>essential singularity</u>.

<u>Example</u>:

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

So, $z=0$ is a <u>removable singularity</u>.

<u>Example</u>:

$$f(z) = \frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

So, the principal part is $\frac{1}{z}$ and $z=0$ is a simple ~~pole~~.

## Zeros

$z_0$ is a zero of $f(z)$ if $f(z_0)=0$.

$z_0$ is a zero of order $n$ of $f(z)$ if

$f(z_0)=0,\ f'(z_0)=0, \cdots\ f^{(n-1)}(z_0)=0$ but $f^{(n)}(z_0)\neq 0$.

or

$$f(z)=a_n(z-z_0)^n+a_{n+1}(z-z_0)^{n+1}+a_{n+2}(z-z_0)^{n+2}+\cdots$$

$$=(z-z_0)^n\left[a_n+a_{n+1}(z-z_0)+a_{n+2}(z-z_0)^2+\cdots\right]$$

Example: What is the order of $f(z)=z\sin z^2$'s zero at $z=0$.

$$f(z)=z\sin z^2=z\left[z^2-\frac{z^6}{3!}+\frac{z^{10}}{5!}-\cdots\right]$$

$$=z^3-\frac{z^7}{3!}+\frac{z^{11}}{5!}-\cdots$$

$$=z^3\left[1-\frac{z^4}{3!}+\frac{z^8}{5!}-\cdots\right]$$

So, $z=0$ is a zero of order 3.

Theorem: I functions $f(z)$ and $g(z)$ are analytic at $z=z_0$ and $f(z)$ has a zero of order $n$ at $z=z_0$ and $g(z_0)\neq 0$ then the function $F(z)=\dfrac{g(z)}{f(z)}$ has a pole of order $n$ at $z=z_0$.

Example: What ~~are~~ is the order of poles of

$$f(z) = \frac{3z-1}{z^2+2z+5} = \frac{3z-1}{(z+1-2i)(z+1+2i)}$$

So, a simple pole at $z=-1+2i$

and a simple pole at $z=1-2i$.

Example: order of poles of

$$f(z) = \frac{1}{1-e^z}$$

$$1-e^z = 0 \Rightarrow e^z = 1 \Rightarrow e^z = e^{in2\pi}$$

$$\Rightarrow z = 2n\pi i \qquad n = 0, \pm 1, \pm 2, \ldots$$

are simple poles.