× Lecture 6, Oct. 13, 2010

Performance Evaluation of different detection schemes:

The performance of a detection scheme $\tilde{\delta}$ can be judged by its probability of miss

$$P_M(\tilde{\delta}) = P_1(\tilde{\delta} \text{ chooses } H_0)$$

and the probability of false alarm

$$P_F(\tilde{\delta}) = P_0(\tilde{\delta} \text{ choose } H_1).$$

For a typical likelihood ratio test

$$\tilde{\delta}_T(y) = \begin{cases} 1 & > \\ \gamma & T(y) = \tau \\ 0 & < \end{cases}$$

we have

$$P_F(\tilde{\delta}_T) = \int \cdots \int_{\{T(y) > \tau\}} p(y_1, \cdots y_n | H_0)\, dy_1 \cdots dy_n$$

$$+ \gamma \int \cdots \int_{\{T(y) = \tau\}} p(y_1, \cdots y_n | H_0)\, dy_1 \cdots dy_n$$

$$P_F(\tilde{\delta}_T) = P[T(Y) > \tau \mid H_0] + \gamma P[T(Y) = \tau \mid H_0]$$

$$= [1 - F_{T,0}(\tau)] + \gamma [F_{T,0}(\tau) - \lim_{\sigma \to \tau^{-1}} F_{T,0}(\sigma)]$$

and

$$P_M(\tilde{\delta}_T) = P(T(Y) < \tau \mid H_1) + (1-\gamma) P(T(Y) = \tau \mid H_1)$$

$$= P(T(Y) \leq \tau \mid H_1) - \gamma P(T(Y) = \tau \mid H_1)$$

$$= F_{T,1}(\tau) - \gamma [F_{T,1}(\tau) - \lim_{\sigma \to \tau^{-1}} F_{T,1}(\sigma)]$$

where $F_{T,0}(\tau)$ and $F_{T,1}(\tau)$ are the CDF of T under H_0 and H_1, respectively.

Chernoff bound

Finding exact value of $P_F(\tilde{\delta})$ and $P_M(\tilde{\delta})$ is not easy, if possible at all, for most cases of interest. The alternative is to use bounds to assess the performance of detection schemes. Chernoff bound is one such bound.

Let's first prove a useful inequlity that we use to derive the Chernoff bound.

Markov inequality

For a positive random variable, i.e., one such that

$$P(X=x)=0 \qquad \forall x<0$$

we have

$$P(X\geq a)\leq \frac{1}{a}E[X]$$

Proof:

$$aP(X\geq a)=a\int_a^\infty p(x)\,dx \leq \int_a^\infty xp(x)\,dx$$

$$\leq \int_0^\infty xp(x)\,dx)=E[X]$$

So

$$P(X\geq a)\leq \frac{1}{a}E[X]$$

Now, let's apply the Markov inequality to $P_{\cancel{F}}(\tilde{\delta}_T)$:

$$P_F(\tilde{\delta}_T)\leq P(T(Y)\geq\tau\,|\,H_0)=P(e^{sT(Y)}\geq e^{s\tau}\,|\,H_0)$$

(since $\gamma<1$), for all $s>0$

$$P_F(\tilde{\delta}_T)\leq P(e^{sT(Y)}\geq e^{s\tau}\,|\,H_0)$$

$$\leq e^{-s\tau}E[e^{sT(Y)}\,|\,H_0]$$

or

$$P_F(\tilde{\delta}_T) \leq \exp[-s\tau + \mu_{T,0}(s)] \qquad (A)$$

where $\mu_{T,0}(s)$ is the cumulant generating function of $T(Y)$ under H_0, defined as

$$\mu_{T,0}(s) = \log\left(E\{e^{sT(Y)} | H_0\}\right).$$

Similarly, since $\gamma \geq 0$

$$P_M(\tilde{\delta}_T) \leq P[T(Y) \leq \tau | H_1] = P[e^{tT(y)} \geq e^{t\tau} | H_1]$$

$$\leq \exp[-t\tau + \mu_{T,1}(t)] \quad (B)$$

for all $t < 0$ wher $\mu_{T,1}(t)$ is the cumulant generating function of $T(Y)$ under H_1

$$\mu_{T,1}(t) = \log\left(E\{e^{tT(Y)} | H_1\}\right)$$

The bounds found (A & B) can be made tighter by minimizing over $s > 0$ and $t < 0$, respectively.

For a likelihood ratio test, we write:

$$T(y) = \log L(y) = \log \frac{p(y|H_1)}{p(y|H_0)}$$

So, we have:

$$\mu_{T,0}(s) = \log\left(\int_{\Gamma} e^{sL(y)} p(y|H_0)\,dy \right)$$

$$= \log\left(\int_{\Gamma} (L(y))^{s} p(y|H_0)\,dy \right)$$

and

$$\mu_{T,1}(t) = \log\left(\int_{\Gamma} (L(y))^{t} p(y|H_1)\,dy \right)$$

$$= \log\left(\int_{\Gamma} (L(y))^{t+1} p(y|H_0)\,dy \right)$$

$$= \mu_{T,0}(t+1)$$

Thus,

$$P_M(\tilde{\delta}_T) \leq \exp\left\{ (1-s)\tau + \mu_{T,0}(s) \right\} \quad s<1$$

So, we need to minimize $[\mu_{T,0}(s) - s\tau]$ over $s>0$ and also over $s<1$. If these two minima fall in the range $0<s<1$ then we have a sing s making the bound tight i.e., if

$$\arg\left\{ \min_{s<1} [\mu_{T,0}(s) - s\tau] \right\} > 0$$

and

$$\arg\left\{\min_{s>0}\left[\mu_{T,0}(s)-s\tau\right]\right\}<1$$

It can be shown that $[\mu_{T,0}(s) - s\tau]$ is a convex function of s. It is also easy to show that

$$\mu'_{T,0}(j) = E[\log L(y) \mid H_j] \qquad j=0,1$$

where $\mu'_{T,0}(j) = \mu'_{T,0}(s)\big|_{s=j}$

and $\mu'_{T,0}(s) = \frac{d}{ds}\mu_{T,0}(s)$.

Since minimum occurs when

$$\mu'_{T,0}(s) = \tau$$

and $\mu'_{T,0}(s)$ is an increasing function of s (due to the convexity of $[\mu_{T,0}(s)-s\tau]$) then a minimum point in $s\in(0,1)$ exists if:

$$\mu_0 \triangleq E[\log L(y)|H_0] < \tau < E[\log L(y)|H_1] \triangleq \mu_1$$

In such a case:

$$\boxed{\begin{aligned} &P_F(\tilde{\delta}_T) \leq \exp[\mu_{T,0}(s_0) - s_0\,\mu'_{T,0}(s_0)] \\ &\text{and} \\ &P_M(\tilde{\delta}_T) \leq \exp[\mu_{T,0}(s_0) + (1-s_0)\,\mu'_{T,0}(s_0)] \end{aligned}}$$

with

$$\mu_0 < \mu'_{T,0}(s_0) = \tau < \mu_1 .$$

The above equation are called the <u>Chernoff bound</u>.