An Introduction to Signal Detection and Estimation - Second Edition
Chapter II: Selected Solutions

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Exercise 2:

The likelihood ratio is given by

$$L(y) = \frac{3}{2(y+1)}, \quad 0 \le y \le 1.$$

a. With uniform costs and equal priors, the critical region for minimum Bayes error is given by $\{y \in [0,1] | L(y) \ge 1\} = \{y \in [0,1] | 3 \ge 2(y+1)\} = [0, 1/2]$. Thus the Bayes rule is given by

$$\delta_B(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1/2 \\ 0 & \text{if } 1/2 < y \le 1 \end{cases}.$$

The corresponding minimum Bayes risk is

$$r(\delta_B) = \frac{1}{2} \int_0^{1/2} \frac{2}{3}(y+1)dy + \int_{1/2}^{1} dy = \frac{11}{24}.$$

b. With uniform costs, the least-favorable prior will be interior to $(0,1)$, so we examine the conditional risks of Bayes rules for an equalizer condition. The critical region for the Bayes rule δ_{π_0} is given by

$$\Gamma_1 = \left\{ y \in [0,1] \middle| L(y) \ge \frac{\pi_0}{1-\pi_0} \right\} = [0, \tau'],$$

where

$$\tau' = \begin{cases} 1 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ \frac{1}{2}\left(\frac{3}{\pi_0} - 5\right) & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 0 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases}.$$

Thus, the conditional risks are:

$$R_0(\delta_{\pi_0}) = \int_0^{\tau'} \frac{2}{3}(y+1)dy = \begin{cases} 1 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ \frac{2\tau'}{3}\left(\frac{\tau'}{2}+1\right) & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 0 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases},$$

and

$$R_1(\delta_{\pi_0}) = \int_{\tau'}^1 dy = \begin{cases} 0 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ 1-\tau' & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 1 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases}.$$

By inspection, a minimax threshold τ'_L is the solution to the equation

$$\frac{2\tau'_L}{3}\left(\frac{\tau'_L}{2}+1\right) = 1-\tau'_L,$$

which yields $\tau'_L = (\sqrt{37}-5)/2$. The minimax risk is the value of the equalized conditional risk; i.e., $V(\pi_L) = 1-\tau'_L$.

c. The Neyman-Pearson test is given by

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } \frac{3}{2(y+1)} > \eta \\ \gamma_0 & \text{if } \frac{3}{2(y+1)} = \eta \\ 0 & \text{if } \frac{3}{2(y+1)} < \eta \end{cases},$$

where η and γ_0 are chosen to give false-alarm probability α. Since $L(y)$ is monotone decreasing in y, the above test is equivalent to

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } y < \eta' \\ \gamma_0 & \text{if } y = \eta' \\ 0 & \text{if } y > \eta' \end{cases},$$

where $\eta' = \frac{3}{2\eta} - 1$. Since Y is a continuous random variable, we can ignore the randomization. Thus, the false-alarm probability is:

$$P_F(\delta_{NP}) = P_0(Y < \eta') = \int_0^{\eta'} \frac{2}{3}(y+1)dy = \begin{cases} 0 & \text{if } \eta' \le 0 \\ \frac{2\eta'}{3}\left(\frac{\eta'}{2}+1\right) & \text{if } 0 < \eta' < 1 \\ 1 & \text{if } \eta' \ge 1 \end{cases}.$$

The threshold for $P_F(\delta_{NP}) = \alpha$ is the solution to

$$\frac{2\eta'}{3}\left(\frac{\eta'}{2}+1\right) = \alpha,$$

which is $\eta' = \sqrt{1+3\alpha}-1$. So, an α-level Neyman-Pearson test is

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } y \le \sqrt{1+3\alpha}-1 \\ 0 & \text{if } y > \sqrt{1+3\alpha}-1 \end{cases}.$$

The detection probability is

$$P_D(\delta_{NP}) = \int_0^{\eta'} dy = \eta' = \sqrt{1+3\alpha}-1, \quad 0 < \alpha < 1.$$

Exercise 4:

Here the likelihood ratio is given by

$$L(y) = \sqrt{\frac{2}{\pi}} e^{y - \frac{y^2}{2}} \equiv \sqrt{\frac{2e}{\pi}} e^{-\frac{(y-1)^2}{2}}, \quad y \geq 0.$$

a. Thus, Bayes critical regions are of the form

$$\Gamma_1 = \left\{ y \geq 0 \middle| (y-1)^2 \leq \tau' \right\},$$

where $\tau' = -\sqrt{\frac{\pi}{2e}} \log\left(\frac{\pi_0}{1-\pi_0}\right)$. There are three cases:

$$\Gamma_1 = \begin{cases} \phi & \text{if } \tau' < 0 \\ \left[1-\sqrt{\tau'}, 1+\sqrt{\tau'}\right] & \text{if } 0 \leq \tau' \leq 1 \\ \left[0, 1+\sqrt{\tau'}\right] & \text{if } \tau' > 1 \end{cases}.$$

The condition $\tau' < 0$ is equivalent to $\pi_0' < \pi_0 \leq 1$, where $\pi_0' = \frac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}}$; the condition $0 \leq \tau' \leq 1$ is equivalent to $\pi_0'' \leq \pi_0 \leq \pi_0'$, where $\pi_0'' = \frac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}}$; and the condition $\tau' > 1$ is equivalent to $0 \leq \pi_0 \ < \pi_0''$.

The minimum Bayes risk $V(\pi_0)$ can be calculated for the three regions:

$$V(\pi_0) = 1 - \pi_0, \quad \pi_0' < \pi_0 \leq 1,$$

$$V(\pi_0) = \pi_0 \int_{1-\sqrt{\tau'}}^{1+\sqrt{\tau'}} e^{-y} dy + (1-\pi_0)\sqrt{\frac{2}{\pi}} \left[\int_0^{1-\sqrt{\tau'}} e^{-\frac{y^2}{2}} dy + \int_{1+\sqrt{\tau'}}^{\infty} e^{-\frac{y^2}{2}} dy \right], \quad \pi_0'' \leq \pi_0 \leq \pi_0',$$

and

$$V(\pi_0) = \pi_0 \int_0^{1+\sqrt{\tau'}} e^{-y} dy + (1-\pi_0)\sqrt{\frac{2}{\pi}} \int_{1+\sqrt{\tau'}}^{\infty} e^{-\frac{y^2}{2}} dy, \quad 0 \leq \pi_0 \ < \pi_0''.$$

b. The minimax rule can be found by equating conditional risks. Investigation of the above shows that this equality occurs in the intermediate region $\pi_0'' \leq \pi_0 \leq \pi_0'$, and thus corresponds to a threshold $\tau_L' \in (0,1)$ solving

$$e^{\sqrt{\tau_L'}} - e^{-\sqrt{\tau_L'}} = 2e(1 + \Phi(1+\sqrt{\tau_L'}) - \Phi(1-\sqrt{\tau_L'})).$$

The minimax risk is then either of the equal conditional risks; e.g.,

$$V(\pi_L) = e^{-1+\sqrt{\tau_L'}} - e^{-1-\sqrt{\tau_L'}}.$$

c. Here, randomization is unnecessary, and the Neyman-Pearson critical regions are of the form

$$\Gamma_1 = \left\{ y \geq 0 \middle| (y-1)^2 \leq \eta' \right\},$$

where $\eta' = -\sqrt{\frac{\pi}{2e}}\log(\eta)$. There are three cases:

$$\Gamma_1 = \begin{cases} \phi & \text{if } \eta' < 0 \\ \left[1-\sqrt{\eta'}, 1+\sqrt{\eta'}\right] & \text{if } 0 \le \eta' \le 1 \\ \left[0, 1+\sqrt{\eta'}\right] & \text{if } \eta' > 1 \end{cases}.$$

The false-alarm probability is thus:

$$P_F(\delta_{NP}) = 0, \quad \eta' < 0$$

$$P_F(\delta_{NP}) = \int_{1-\sqrt{\eta'}}^{1+\sqrt{\eta'}} e^{-y}dy = e^{-1+\sqrt{\eta'}} - e^{-1-\sqrt{\eta'}} = \frac{2}{e}\sinh\left(\sqrt{\eta'}\right), \quad 0 \le \eta' \le 1,$$

and

$$P_F(\delta_{NP}) = \int_0^{1+\sqrt{\eta'}} e^{-y}dy = 1 - e^{-1-\sqrt{\eta'}}, \quad \eta' > 1.$$

From this we see that the threshold for $\alpha-$level NP testing is

$$\eta' = \begin{cases} \left[\sinh^{-1}\left(\frac{\alpha e}{2}\right)\right]^2 & \text{if } 0 < \alpha \le 1 - e^{-2} \\ \left[1 + \log(1-\alpha)\right]^2 & \text{if } 1 - e^{-2} < \alpha < 1 \end{cases}.$$

The detection probability is thus

$$P_D(\delta_{NP}) = 2\left[\Phi\left(1+\sqrt{\eta'}\right) - \Phi\left(1-\sqrt{\eta'}\right)\right]$$

$$= 2\left[\Phi\left(1+\sinh^{-1}\left(\frac{\alpha e}{2}\right)\right) - \Phi\left(1-\sinh^{-1}\left(\frac{\alpha e}{2}\right)\right)\right], \quad 0 < \alpha \le 1 - e^{-2},$$

and

$$P_D(\delta_{NP}) = 2\left[\Phi\left(1+\sqrt{\eta'}\right) - \frac{1}{2}\right] = 2\left[\Phi\left(2+\log(1-\alpha)\right) - \frac{1}{2}\right], \quad 1 - e^{-2} < \alpha \le 1.$$

Exercise 6 a & b:

Here we have $p_0(y) = p_N(y+s)$ and $p_1(y) = p_N(y-s)$, which gives

$$L(y) = \frac{1+(y+s)^2}{1+(y-s)^2}.$$

a. With equal priors and uniform costs, the critical region for Bayes testing is $\Gamma_1 = \{L(y) \ge 1\} = \{1+(y+s)^2 \ge 1+(y-s)^2\} = \{2sy \ge -2sy\} = [0,\infty)$. Thus, the Bayes test is

$$\delta_B(y) = \begin{cases} 1 & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}$$

The minimum Bayes risk is then

$$r(\delta_B) = \frac{1}{2}\int_0^\infty \frac{1}{\pi\left[1+(y+s)^2\right]}dy + \frac{1}{2}\int_{-\infty}^0 \frac{1}{\pi\left[1+(y-s)^2\right]}dy = \frac{1}{2} - \frac{\tan^{-1}(s)}{\pi}.$$

b. Because of the symmetry of this problem with uniform costs, we can guess that 1/2 is the least-favorable prior. To confirm this, we can check that this answer from Part a gives an equalizer rule:

$$R_0(\delta_{1/2}) = \int_0^\infty \frac{1}{\pi\left[1+(y+s)^2\right]}dy = \frac{1}{2}\int_{-\infty}^0 \frac{1}{\pi\left[1+(y-s)^2\right]}dy = R_1(\delta_{1/2}).$$

Exercise 7:

a. The densities under the two hypotheses are:

$$p_0(y) = p(y) = e^{-y}, \quad y > 0,$$

and

$$p_1(y) = \int_{-\infty}^\infty p(y-s)p(y)ds = \int_0^y e^{s-y}e^{-s}ds = ye^{-y}, \quad y > 0.$$

Thus, the likelihood ratio is

$$L(y) = \frac{p_1(y)}{p_0(y)} = y, \quad y > 0.$$

b. Randomization is irrelevant here, so the false-alrm probability for threshold η is

$$P_F(\delta_{NP}) = P_0(Y > \eta) = e^{-\eta},$$

which gives the threshold $\eta = -\log\alpha$, for $\alpha-$level Neyman-Pearson testing. The corresponding detection probability is

$$P_D(\delta_{NP}) = P_1(Y > \eta) = \int_\eta^\infty ye^{-y}dy = (\eta+1)e^{-\eta} = \alpha(1-\log\alpha), \quad 0 < \alpha < 1.$$

c. Here the densities under the two hypotheses become:

$$p_0(y) = \prod_{k=1}^n p(y_k) = \prod_{k=1}^n e^{-y_k}, \quad 0 < \min\{y_1, y_2, \ldots, y_n\},$$

and

$$p_1(y) = \int_{-\infty}^\infty \left[\prod_{k=1}^n p(y_k - s)\right] p(s)ds = \int_0^{\min\{y_1,y_2,\ldots,y_n\}} \left[\prod_{k=1}^n e^{s-y_k}\right] e^{-s}ds$$

$$= \frac{p_0(y)}{n-1}\left[e^{(n-1)\min\{y_1,y_2,\ldots,y_n\}} - 1\right], \quad 0 < \min\{y_1, y_2, \ldots, y_n\}.$$

Thus, the likelihood ratio is

$$L(y) = \frac{1}{n-1}\left[e^{(n-1)\min\{y_1,y_2,\ldots,y_n\}} - 1\right], \quad 0 < \min\{y_1, y_2, \ldots, y_n\}.$$

d. The false-alarm probability incurred by comparing $L(y)$ from Part c to a threshold η is

$$P_F(\delta_{NP}) = P_0(L(Y) > \eta) = P_0\left(\min\{Y_1, Y_2, \ldots, Y_n\} > \eta' \equiv \frac{\log((n-1)\eta + 1)}{n-1}\right)$$

$$= P_0(\bigcap_{k=1}^{n}(Y_k > \eta')) = \prod_{k=1}^{n} P_0(Y_k > \eta') = \prod_{k=1}^{n} e^{-\eta'} = e^{-n\eta'},$$

from which we have $\eta' = -\frac{1}{n}\log\alpha$, or, equivalently,

$$\eta = \frac{e^{(n-1)\eta'} - 1}{n-1} = \frac{\alpha^{-(n-1)/n} - 1}{n-1}.$$

Exercise 15

a. The LMP test is

$$\tilde{\delta}_{lo}(y) = \begin{cases} 1 & \text{if } \frac{\partial p_\theta(y)}{\partial\theta}|_{\theta=0} > \eta p_0(y) \\ \gamma, & \text{if } \frac{\partial p_\theta(y)}{\partial\theta}|_{\theta=0} = \eta p_0(y) \\ 0 & \text{if } \frac{\partial p_\theta(y)}{\partial\theta}|_{\theta=0} < \eta p_0(y) \ . \end{cases}$$

we have

$$\frac{\frac{\partial p_\theta(y)}{\partial\theta}|_{\theta=0}}{p_0(y)} = sgn(y) \ ;$$

thus

$$\tilde{\delta}_{lo}(y) = \begin{cases} 1 & \text{if } sgn(y) > \eta \\ \gamma, & \text{if } sgn(y) = \eta \\ 0 & \text{if } sgn(y) < \eta \ . \end{cases}$$

To set the threshold η, we consider

$$P_0(sgn(Y) > \eta) = \begin{cases} 0 & \text{if } \eta \ge 1 \\ 1/2 & \text{if } -1 \le \eta < 1 \\ 1 & \text{if } \eta < -1 \ . \end{cases}$$

This implies that

$$\eta = \begin{cases} 1 & \text{if } 0 < \alpha < 1/2 \\ -1 & \text{if } 1/2 \le \alpha < 1 \ . \end{cases}$$

The randomization is

$$\gamma = \frac{\alpha - P_0(sgn(Y) > \eta)}{P_0(sgn(Y) - \eta)} = \begin{cases} 2\alpha & \text{if } 0 < \alpha < 1/2 \\ 2\alpha - 1 & \text{if } 1/2 \le \alpha < 1 \;. \end{cases}$$

The LMP test is thus

$$\delta_{lo}(y) = \begin{cases} 2\alpha & \text{if } y > 0 \\ 0 & \text{if } y \le 0 \end{cases}$$

for $0 < \alpha < 1/2$; and it is

$$\delta_{lo}(y) = \begin{cases} 1 & \text{if } y \ge 0 \\ 2\alpha - 1 & \text{if } y < 0 \end{cases}$$

for $1/2 \le \alpha < 1$.

For fixed $\theta > 0$, the detection probability is

$$P_D(\tilde{\delta}_{lo};\theta) = P_\theta(sgn(Y) > \eta) + \gamma P_\theta(sgn(Y) = \eta)$$

$$= \begin{cases} 2\alpha \int_0^\infty \frac{1}{2} e^{-|y-\theta|}dy & \text{if } 0 < \alpha < 1/2 \\ \int_0^\infty \frac{1}{2} e^{-|y-\theta|}dy + (2\alpha - 1)\int_{-\infty}^0 \frac{1}{2} e^{-|y-\theta|}dy & \text{if } 1/2 \le \alpha < 1 \end{cases}$$

$$= \begin{cases} \alpha(2 - e^{-\theta}) & \text{if } 0 < \alpha < 1/2 \\ 1 + (\alpha - 1)e^{-\theta} & \text{if } 1/2 \le \alpha < 1 \end{cases} .$$

b. For fixed θ, the NP critical region is

$$\Gamma_\theta = \{|y| - |y - \theta| > \eta'\}$$

$$= \begin{cases} (-\infty, \infty) & \text{if } \eta' < -\theta \\ ((\frac{\eta'+\theta}{2}), \infty) & \text{if } -\theta \le \eta' \le \theta \\ \phi & \text{if } \eta' > \theta \;, \end{cases}$$

from which

$$P_0(\Gamma_\theta) = \begin{cases} 1 & \text{if } \eta' < -\theta \\ \frac{1}{2} e^{-(\eta'+\theta)/2} & \text{if } -\theta \le \eta' \le \theta \\ 0 & \text{if } \eta' > \theta \;. \end{cases}$$

Clearly, we must know θ to set η', and thus the NP critical region depends on θ. This implies that there is no UMP test.

The generalized likelihood ratio test uses this statistic:

$$\sup_{\theta > 0} e^{|y| - |y-\theta|} = \exp\{\sup_{\theta > 0}(|y| - |y - \theta|)\}$$

$$= \begin{cases} 1 & \text{if } y < 0 \\ e^y & \text{if } y \ge 0 \end{cases} .$$

Exercise 16:

We have M hypotheses $H_0, H_1 \ldots, H_{M-1}$, where Y has distribution P_i and density p_i under hypothesis H_i. A decision rule δ is a partition of the observation set Γ into regions $\Gamma_0, \Gamma_1, \ldots, \Gamma_{M-1}$, where δ chooses hypothesis H_i when we observe $y \in \Gamma_i$. Equivalently, a decision rule can be viewed as a mapping from Γ to the set of decisions $\{0, 1, \ldots, M-1\}$, where $\delta(y)$ is the index of the hypothesis accepted when we observe $Y = y$.

On assigning costs C_{ij} to the acceptance of H_i when H_j is true, for $0 \leq i, j \leq (M-1)$, we can define *conditional risks*, $R_j(\delta), j = 0, 1, \ldots, M-1$, for a decision rule δ, where $R_j(\delta)$ is the conditional expected cost given that H_j is true. We have

$$R_j(\delta) = \sum_{i=0}^{M-1} C_{ij} P_j(\Gamma_i).$$

Assuming priors $\pi_j = P(H_j occurs), j = 0, 1, \ldots, M-1$, we can define an overall average risk or *Bayes risk* as

$$r(\delta) = \sum_{j=0}^{M-1} \pi_j R_j(\delta).$$

A *Bayes rule* will minimize the Bayes risk.

We can write

$$r(\delta) = \sum_{j=0}^{M-1} \sum_{i=0}^{M-1} \pi_j C_{ij} P_j(\Gamma_i) = \sum_{i=0}^{M-1} \left[\sum_{j=0}^{M-1} \pi_j C_{ij} P_j(\Gamma_i) \right]$$

$$= \sum_{i=0}^{M-1} \left[\sum_{j=0}^{M-1} \pi_j C_{ij} \int_{\Gamma_i} p_j(y) \mu(dy) \right] = \sum_{i=0}^{M-1} \int_{\Gamma_i} \left[\sum_{j=0}^{M-1} \pi_j C_{ij} p_j(y) \right] \mu(dy).$$

Thus, by inspection, we see that the Bayes rule has decision regions given by

$$\Gamma_i = \left\{ y \in \Gamma \,\middle|\, \sum_{j=0}^{M-1} \pi_j C_{ij} p_j(y) = \min_{0 \leq k \leq M-1} \sum_{j=0}^{M-1} \pi_j C_{kj} p_j(y) \right\}.$$

Exercise 19:

a. The likelihood ratio is given by

$$L(y) = \frac{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(y_k - \mu_1)^2 / 2\sigma_1^2}}{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(y_k - \mu_0)^2 / 2\sigma_0^2}}$$

$$= \left(\frac{\sigma_0}{\sigma_1} \right)^n e^{\frac{n}{2} \left(\frac{\mu_0^2}{\sigma_0^2} - \frac{\mu_1^2}{\sigma_1^2} \right)} e^{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} \right) \sum_{k=1}^{n} y_k^2} e^{\left(\frac{\mu_1}{\sigma_1^2} - \frac{\mu_0}{\sigma_0^2} \right) \sum_{k=1}^{n} y_k},$$

which shows the structure indicated.

b. If $\mu_1 = \mu_0 \equiv \mu$ and $\sigma_1^2 > \sigma_0^2$, then the Neyman-Pearson test operates by comparing the quantity $\sum_{k=1}^n (y_k - \mu)^2$ to a threshold, choosing H_1 if the threshold is exceeded and H_0 otherwise. Alternatively, if $\mu_1 > \mu_0$ and $\sigma_1^2 = \sigma_0^2$, then the NP test compares $\sum_{k=1}^n y_k$ to a threshold, again choosing H_1 when the threshold is exceeded. Note that, in the first case, the test statistic is *quadratic* in the observations, and in the second case it is *linear*.

c. For $n = 1$, $\mu_1 = \mu_0 \equiv \mu$ and $\sigma_1^2 > \sigma_0^2$,, the NP test is of the form

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } (y_1 - \mu)^2 \geq \eta' \\ 0 & \text{if } (y_1 - \mu)^2 < \eta' \end{cases} ,$$

where $\eta' > 0$ is an appropriate threshold. We have

$$P_F(\delta_{NP}) = P_0((Y_1 - \mu)^2 > \eta') = 1 - P_0(-\sqrt{\eta'} \leq Y_1 - \mu \leq \sqrt{\eta'})$$

$$= 1 - \Phi\left(\frac{\sqrt{\eta'}}{\sigma_0}\right) + \Phi\left(-\frac{\sqrt{\eta'}}{\sigma_0}\right) = 2\left[1 - \Phi\left(\frac{\sqrt{\eta'}}{\sigma_0}\right)\right].$$

Thus, for size α we set

$$\eta' = \left[\sigma_0 \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\right]^2,$$

and the detection probability is

$$P_D(\delta_{NP}) = 1 - P_1(-\sqrt{\eta'} \leq Y_1 - \mu \leq \sqrt{\eta'}) = 2\left[1 - \Phi\left(\frac{\sqrt{\eta'}}{\sigma_1}\right)\right]$$

$$= 2\left[1 - \Phi\left(\frac{\sigma_0}{\sigma_1}\Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\right)\right], \quad 0 < \alpha < 1.$$