An Introduction to Signal Detection and Estimation - Second Edition
Chapter III: Selected Solutions

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Exercise 1:

Let $\{h_{k,l}\}$ denote the impulse response of a general discrete-time linear filter. The output at time n due to the input signal is $\sum_{l=1}^{n} h_{n,l} s_l$, and that due to noise is $\sum_{l=1}^{n} h_{n,l} N_l$. Thus the output SNR at time n is

$$SNR = \frac{|\sum_{l=1}^{n} h_{n,l} s_l|^2}{E\{(\sum_{l=1}^{n} h_{n,l} N_l)^2} = \frac{|\underline{h}_n^T \underline{s}|^2}{\underline{h}_n^T \mathbf{\Sigma}_N \underline{h}_n}$$

where $\underline{h}_n = (h_{n,1}, h_{n,2}, \ldots, h_{n,n})^T$.

Since $\mathbf{\Sigma}_N > 0$, we can write $\mathbf{\Sigma}_N = \mathbf{\Sigma}_N^{1/2} \mathbf{\Sigma}_N^{1/2}$ when $\mathbf{\Sigma}_N^{1/2}$ is invertible and symmetric. Thus,

$$SNR = \frac{|(\mathbf{\Sigma}_N^{1/2} \underline{h}_n)^T \mathbf{\Sigma}_N^{-1/2} \underline{s}|^2}{||\mathbf{\Sigma}_N^{1/2} \underline{h}_n||^2}$$

By the Schwarz Inequality ($|\underline{x}^T \underline{y}| \leq ||x|| \; ||y||$, we have

$$SNR \leq ||\mathbf{\Sigma}_N^{1/2} \underline{s}||^2$$

with equality if and only if $\mathbf{\Sigma}_N^{1/2} \underline{h}_n = \lambda \mathbf{\Sigma}_N^{-1/2} \underline{s}$ for a constant λ. Thus, max SNR occurs when $\underline{h}_n = \lambda \mathbf{\Sigma}_N^{-1} \underline{s}$. The constant λ is arbitrary (it does not affect SNR), so we can take $\lambda = 1$, which gives the desired result.

Exercise 3:

a. From Exercise 15 of Chapter II, the optimum test here has critical regions:

$$\Gamma_k = \{\underline{y} \in R^n \Big| p_k(\underline{y}) = \max_{0 \leq l \leq M-1} p_l(\underline{y})\}.$$

Since p_l is the $N(\underline{s}_l, \sigma^2\mathbf{I})$ density, this reduces to

$$\Gamma_k = \{\underline{y} \in R^n \left| \|\underline{y} - \underline{s}_k\|^2 = \min_{0\le l\le M-1} \|y - s_l\|^2 \right\}$$

$$= \{\underline{y} \in R^n \left| \underline{s}_k^T \underline{y} = \max_{0\le l\le M-1} \underline{s}_l^T \underline{y} \right\} \ .$$

b. We have

$$P_e = \frac{1}{M}\sum_{k=0}^{M} P_k(\Gamma_k^c),$$

and

$$P_k(\Gamma_k^c) = 1 - P_k(\Gamma_k) = 1 - P_k(\max_{0\le l\ne k\le M-1} \underline{s}_l^T \underline{Y} < \underline{s}_k^T \underline{Y})$$

Due to the assumed orthogonality of $\underline{s}_1, \cdots, \underline{s}_n$, it is straighforward to show that, under H_k, $\underline{s}_1^T\underline{Y}, \underline{s}_2^T\underline{Y}, \cdots, \underline{s}_n^T\underline{Y}$, are independent Gaussian random variable with variances $\sigma^2\|s_1\|^2$, and with means zero for $l \ne k$ and mean $\|s_1\|^2$ for $l = k$. Thus

$$P_k(\max_{0\le l\ne k\le M-1} \underline{s}_l^T \underline{Y} < \underline{s}_k^T \underline{Y})$$

$$= \frac{1}{\sqrt{2\pi}\sigma\|s_1\|}\int_{-\infty}^{\infty} P_k(\max_{0\le l\ne k\le M-1} \underline{s}_L^T \underline{Y} < z) \ e^{-(z-\|s_1\|^2)/2\sigma^2\|s_1\|^2} \ dz$$

Now

$$P_k(\max_{0\le l\ne k\le M-1} \underline{s}_L^T \underline{Y} < z) = P_k(\bigcap_{0\le l\ne k\le M-1} \{\underline{s}_l^T \underline{Y} < z\})$$

$$= \prod_{0\le l\ne k\le M-1} P_k(\underline{s}_l^T \underline{Y} < z)$$

$$= \left[\Phi(\frac{z}{\sigma\|s_1\|})\right]^{M-1} .$$

Combining the above and setting $x = z/\sigma\|s_1\|$ yields

$$1 - P_k(\Gamma_k) = \frac{1}{\sqrt{2\pi}} \quad \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} \ e^{-(x-d)^{\frac{2}{2}}} dx, k = 0, \cdots, M-1 \ ,$$

and the desired expression for P_e follows.

Exercise 6:

Since $\underline{Y} \sim N(\underline{\mu}, \mathbf{\Sigma})$, it follows that $\hat{Y}_k$ is linear in $Y_1, \cdots, Y_{k-1}$, and that $\hat{\sigma}^2_{Y_k}$ does not depend on $Y_1, \cdots, Y_{k-1}$. Thus, $\underline{I}$ is a linear transformation of $\underline{Y}$ and is Gaussian. We need only show that $E\{\underline{I}\} = \underline{0}$ and $cov(\underline{I}) = \mathbf{I}$.

We have

$$E\{I_k\} = \frac{E\{Y_k\} - E\{\hat{Y}_k\}}{\hat{\sigma_k}} \ .$$

Since $\hat{Y}_k = E\{Y_k|Y_1, \cdots, Y_{k-1}\}$, $E\{\hat{Y}_k\}$ is an iterated expectation of Y_k; hence $E\{Y_k\} = E\{\hat{Y}_k\}$ and $E\{I_k\} = 0, k = 1, \cdots, n$. To see that $cov(\underline{I}) = \mathbf{I}$, note first that

$$\text{Var }(I_k) = E\{I_k^2\} = \frac{E\{(Y_k - \hat{Y}_k)^2\}}{\hat{\sigma}_{Y_k}^2} = \frac{\hat{\sigma}_{Y_k}^2}{\hat{\sigma}_{Y_k}^2} = 1 \ .$$

Now, for $l < k$, we have

$$\begin{aligned} \text{cov }(I_k, I_l) &= E\{I_k I_l\} \\ &= \frac{E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\}}{\hat{\sigma}_{Y_k}\hat{\sigma}_{Y_l}} \ . \end{aligned}$$

Noting that

$$E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = E\{E\{(Y_k - \hat{Y}_k)(Y_l - \hat{Y}_l)|Y_1, \cdots, Y_{k-1}\}\}$$

$$= E\{(E\{Y_k|Y_1, \cdots, Y_{k-1}\} - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = E\{(\hat{Y}_k - \hat{Y}_k)(Y_l - \hat{Y}_l)\} = 0 \ ,$$

we have $cov(I_k, I_l) = 0$ for $l < k$. By symmetry we also have $cov(I_k, I_l) = 0$ for $l > k$, and the desired result follows.

Exercise 7:

a. The likelihood ratio is

$$L(y) = \frac{1}{2}e^{\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y} - d^2/2} + \frac{1}{2}e^{-\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y} - d^2/2}$$

$$= e^{-d^2/2}\cosh \underline{s}^T\mathbf{\Sigma}^{-1}\underline{y} \quad ,$$

which is monotone increasing in the statistic

$$T(\underline{y}) \equiv \left|\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y}\right| .$$

(Here, as usual, $d^2 = \underline{s}^T\mathbf{\Sigma}^{-1}\underline{s}$.) Thus, the Neyman-Pearson test is of the form

$$\tilde{\delta}_{NP}(\underline{y}) = \begin{cases} 1 & \text{if } T(\underline{y}) > \eta \\ \gamma, & \text{if } T(\underline{y}) = \eta \\ 0 & \text{if } T(\underline{y}) < \eta \ . \end{cases}$$

To set the threshold η, we consider

$$P_0(T(\underline{Y}) > \eta) = 1 - P\left(-\eta \le \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \le \eta\right) = 1 - \Phi(\eta/d) + \Phi(-\eta/d) = 2[1 - \Phi(\eta/d)],$$

where we have used the fact that $\underline{s}^T \mathbf{\Sigma}^{-1} \underline{N}$ is Gaussian with zero mean and variance d^2. Thus, the threshold for size α is

$$\eta = d\Phi^{-1}(1-\alpha/2).$$

The randomization is unnecessary.

The detection probability is

$$P_D(\tilde{\delta}_{NP}) = \frac{1}{2}P_1(T(\underline{Y}) > \eta | \Theta = +1) + \frac{1}{2}P_1(T(\underline{Y}) > \eta | \Theta = -1)$$

$$= \frac{1}{2}\left[1 - P\left(-\eta \leq -d^2 + \underline{s}^T \mathbf{\Sigma}^{-1} \underline{N} \leq \eta\right)\right] + \frac{1}{2}\left[1 - P\left(-\eta \leq +d^2 + \underline{s}^T \mathbf{\Sigma}^{-1} \underline{N} \leq \eta\right)\right]$$

$$= 2 - \Phi\left(\Phi^{-1}(1-\alpha/2) + d\right) - \Phi\left(\Phi^{-1}(1-\alpha/2) - d\right).$$

b. Since the likelihood ratio is the average over the distribution of Θ of the likelihood ratio conditioned on Θ, we have

$$L(\underline{y}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{(\theta \underline{s}^T \underline{y} - n\theta^2 \bar{s^2}/2)/\sigma^2} e^{-\theta^2/2\sigma_\theta} d\theta$$

$$= k_1 e^{k_2 |\underline{s}^T \underline{y}|} \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} e^{-(\theta-\mu)^2/2v} d\theta = k_1 e^{k_2 |\underline{s}^T \underline{y}|},$$

where

$$v^2 = \frac{\sigma_\theta^2 n \bar{s^2}}{\sigma_\theta^2 \sigma^2 + n\bar{s^2}},$$

$$\mu = \frac{v^2 \underline{s}^T \underline{y}}{2},$$

$$k_1 = \frac{v}{\sigma_\theta},$$

and

$$k_2 = \frac{v^2}{4}.$$

Exercise 13:

a. In this situation, the problem is that of detecting a Gaussian signal with zero mean and covariance matrix $\mathbf{\Sigma_S} = \text{diag}\{As_1^2, As_2^2, \ldots, As_n^2\}$, in independent i.i.d. Gaussian noise with unit variance; and thus the Neyman-Pearson test is based on the quadratic statistic

$$T(\underline{y}) = \sum_{k=1}^{n} \frac{As_k^2}{As_k^2 + 1} y_k^2.$$

b. Assuming $s_k \neq 0$, for all k, a sufficient condition for a UMP test is that s_k^2 is constant. In this case, an equivalent test statistic is the radiometer $\sum_{k=1}^{n} y_k^2$, which can be given size α without knowledge of A.

c. From Eq. (III.B.110), we see that an LMP test can be based on the statistic

$$T_{lo}(\underline{y}) = \sum_{k=1}^{n} s_k^2 y_k^2.$$

Exercise 15:

Let L_a denote the likelihood ratio conditioned on $A = a$. Then the undonditioned likelihood ratio is

$$L(\underline{y}) = \int_0^\infty L_a(\underline{y}) p_A(a) da = \int_0^\infty e^{-na^2/4\sigma^2} I_0(a^2 \hat{r}/\sigma^2) p_A(a) da,$$

with $\hat{r} \equiv r/A$, where $r = \sqrt{y_c^2 + y_s^2}$ as in Example III.B.5. Note that

$$\hat{r} = \sqrt{\left(\sum_{k=1}^{n} b_k \cos((k-1)\omega_c T_s) y_k\right)^2 + \left(\sum_{k=1}^{n} b_k \sin((k-1)\omega_c T_s) y_k\right)^2},$$

which can be computed without knowledge of A. Note further that

$$\frac{\partial L(\underline{y})}{\partial \hat{r}} = \frac{1}{\sigma^2} \int_0^\infty e^{-na^2/4\sigma^2} a^2 I_0'(a^2 \hat{r}/\sigma^2) p_A(a) da > 0,$$

where we have used the fact that I_0 is monotone increasing in its argument. Thus, $L(\underline{y})$ is monotone increasing in $\hat{r}$, and the Neyman-Pearson test is of the form

$$\tilde{\delta}_{NP}(\underline{y}) = \begin{cases} 1 & \text{if } \hat{r} > \tau' \\ \gamma, & \text{if } \hat{r} = \tau' \\ 0 & \text{if } \hat{r} < \tau' \end{cases}.$$

To get size α we choose τ' so that $P_0(\hat{R} > \tau') = \alpha$. From (III.B.72), we have that

$$P_0(\hat{R} > \tau') = e^{-(\tau')^2/n\sigma^2},$$

from which the size-α desired threshold is $\tau' = \sqrt{-n\sigma^2 \log \alpha}$.

The detection probability can be found by first conditioning on A and then averaging the result over the distribution of A. (Note that we have not used the explicit form of the distribution of A to derive any of the above results.) It follows from (III.B.74) that $P_1(\hat{R} > \tau' | A = a) = Q(b, \tau_0)$ with $b^2 = na^2/2\sigma^2$ and $\tau_0 = \sqrt{2/n}\tau'/\sigma = \sqrt{-2\log\alpha}$. Thus,

$$P_D = \int_0^\infty Q(\frac{a}{\sigma}\sqrt{n/2}, \tau_0) p_A(a) da = \int_0^\infty \int_{\tau_0}^\infty x e^{-(x^2 + na^2/2\sigma^2)/2} I_0(x \frac{a}{\sigma}\sqrt{n/2}) \frac{a}{A_0^2} e^{-a^2/2A_0^2} dx da$$

$$= \int_{\tau_0}^{\infty} x e^{-x^2/2} \int_0^{\infty} \frac{a}{A_0^2} e^{-a^2/2a_0^2} I_0(x \frac{a}{\sigma} \sqrt{n/2}) da dx,$$

where $a_0 = \sqrt{\frac{2A_0^2\sigma^2}{nA_0^2+2\sigma^2}}$. On making the substitution $y = a/a_0$, this integral becomes

$$P_D = \frac{a_0^2}{A_0^2} \int_{\tau_0}^{\infty} x e^{-x^2(1-b_0^2)/2} \int_0^{\infty} y e^{-(y^2+b_0^2x^2)/2} I_0(b_0 xy) dy dx = \frac{a_0^2}{A_0^2} \int_{\tau_0}^{\infty} x e^{-x^2(1-b_0^2)/2} Q(b_0 x, 0) dx,$$

where $b_0^2 = na_0^2/2\sigma^2$. Since $Q(b, 0) = 1$ for any value of b, and since $1 - b_0^2 = a_0^2/A_0^2$, the detection probability becomes

$$P_D = \frac{a_0^2}{A_0^2} \int_{\tau_0}^{\infty} x e^{-x^2(1-b_0^2)/2} dx = e^{-\tau_0^2(1-b_0^2)/2} = \exp(-\frac{\tau_0^2}{2\left(1 + \frac{nA_0^2}{2\sigma^2}\right)}) = \alpha^{x_0},$$

where $x_0 = \frac{1}{1+\frac{nA_0^2}{2\sigma^2}}$.

Exercise 16:

The right-hand side of the given equation is simply the likelihood ratio for detecting a $\mathcal{N}(\underline{0}, \mathbf{\Sigma_S})$ signal in independent $\mathcal{N}(\underline{0}, \sigma^2\mathbf{I})$ noise. From Eq. (III.B.84), this is given by

$$\exp(\frac{1}{2\sigma^2} \underline{y}^T \mathbf{\Sigma_S} (\sigma^2\mathbf{I} + \mathbf{\Sigma_S})^{-1} \underline{y} + \frac{1}{2} \log(|\sigma^2\mathbf{I}|/|\sigma^2\mathbf{I} + \mathbf{\Sigma_S}|)).$$

We thus are looking for a solution $\underline{\hat{S}}$ to the equation

$$2\underline{\hat{S}}^T \underline{y} - \| \underline{\hat{S}} \|^2 = \underline{y}^T \mathbf{\Sigma_S} (\sigma^2\mathbf{I} + \mathbf{\Sigma_S})^{-1} \underline{y} + \sigma^2 \sum_{k=1}^{n} \log(\frac{\sigma^2}{\sigma^2 + \lambda_k}),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{\Sigma_S}$. On completing the square on the left-hand side of this equation, it can be rewritten as

$$\| \underline{\hat{S}} - \underline{y} \|^2 = \| \underline{y} \|^2 - \underline{y}^T \mathbf{\Sigma_S} (\sigma^2\mathbf{I} + \mathbf{\Sigma_S})^{-1} \underline{y} - \sigma^2 \sum_{k=1}^{n} \log(\frac{\sigma^2}{\sigma^2 + \lambda_k})$$

$$\equiv \sigma^2 \left[\underline{y}^T (\sigma^2\mathbf{I} + \mathbf{\Sigma_S})^{-1} \underline{y} - \sum_{k=1}^{n} \log(\frac{\sigma^2}{\sigma^2 + \lambda_k}) \right],$$

which is solved by

$$\underline{\hat{S}} = \underline{y} \pm \frac{\sigma}{\| \underline{v} \|} \left[\underline{y}^T (\sigma^2\mathbf{I} + \mathbf{\Sigma_S})^{-1} \underline{y} - \sum_{k=1}^{n} \log(\frac{\sigma^2}{\sigma^2 + \lambda_k}) \right]^{1/2} \underline{v},$$

for any nonzero vector $\underline{v}$.