Chapter V: Selected Solutions

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Exercise 3:

Since $\underline{s}_k$ depends completely on $\underline{Y}_0^k$, it behaves like a constant when conditional quantities given $\underline{Y}_0^k$ are computed. The only point at which this affects the derivation of the Kalman-Bucy filter is in the time update equations, which now become:

$$\underline{\hat{X}}_{t+1|t} = \mathbf{F}_t \underline{\hat{X}}_{t|t} + E\{\mathbf{\Gamma}_t \underline{s}_t | \underline{Y}_0^t\} = \mathbf{F}_t \underline{\hat{X}}_{t|t} + \mathbf{\Gamma}_t \underline{s}_t,$$

and

$$\mathbf{\Sigma}_{t+1|t} = \mathbf{F}_t \mathbf{\Sigma}_{t|t} \mathbf{F}_t^T + \mathbf{G}_t \mathbf{Q}_t \mathbf{G}_t^T + Cov\left(\mathbf{\Gamma}_t \underline{s}_t | \underline{Y}_0^t\right) = \mathbf{F}_t \mathbf{\Sigma}_{t|t} \mathbf{F}_t^T + \mathbf{G}_t \mathbf{Q}_t \mathbf{G}_t^T.$$

Note that the second of these two equations is the same as when there is no measurement feedback.

The measurement feedback has no effect on the measurement update equations. Although the presence of this feedback may cause the state to be nonGaussian, the joint *conditional* statistics of $\underline{X}_t$ and $\underline{Y}_t$ given $\underline{Y}_0^{t-1}$ are still Gaussian. Thus, the measurement update is unchanged from the case of no measurement feedback since it depends only on this joint Gaussian property and the linearity of measurement equation.

Exercise 4:

There are several ways of approaching this problem. An interesting one is to note that, although $\underline{U}_t$ and $\underline{V}_t$ are dependent, the Gaussian vector $\underline{U}'_t \equiv \underline{U}_t - \mathbf{C}_t \mathbf{R}_t^{-1} \underline{V}_t$ is independent of $\underline{V}_t$. To take advantage of this, we may add the zero quantity $\mathbf{C}_t \mathbf{R}_t^{-1} [\underline{Y}_t - \mathbf{H}_t \underline{X}_t - \underline{V}_t]$ to the t^{th} state input, which yields the equivalent state equation

$$\underline{X}_{t+1} = \mathbf{F}_t \underline{X}_t + \mathbf{G}_t \underline{U}'_t + \mathbf{G}_t \mathbf{C}_t \mathbf{R}_t^{-1} \left(\underline{Y}_t - \mathbf{H}_t \underline{X}_t\right).$$

So, we have an equivalent problem with independent state and measurement noises, but with the measurement feedback term $\mathbf{G}_t \mathbf{C}_t \mathbf{R}_t^{-1} \underline{Y}_t$, and with the new state matrix $\left(\mathbf{F}_t - \mathbf{G}_t \mathbf{C}_t \mathbf{R}_t^{-1} \mathbf{H}_t\right)$. We also have a different correlation matrix for the state input, since

$$Cov(\underline{U}'_t) = \mathbf{Q}_t - \mathbf{C}_t \mathbf{R}_t^{-1} \mathbf{C}_t^T.$$

Applying the result of Exercise 3 and eliminating the measurement update equations yields the given result.

Exercise 6:

a. This result follows by induction on t. We first note that, for $t = j$, the given equality follows by definition. From the state equation we have that, for $k > j$,

$$E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)\underline{X}_{k+1}^T\right\} = E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)(\mathbf{F}_k\underline{X}_k + \mathbf{G}_k\underline{U}_k)^T\right\} = E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)\underline{X}_k^T\right\}\mathbf{F}_k^T,$$

where we use the fact that $\underline{U}_k$ has zero mean and is independent of both $\underline{X}_j$ and $\underline{\hat{X}}_{j|k}$. Now, on applying the recursion for $\underline{\hat{X}}_{j|k}$ to this equation we have

$$E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)\underline{X}_{k+1}^T\right\} = E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k-1}\right)\underline{X}_k^T\right\}\mathbf{F}_k^T - \mathbf{K}_k^a E\left\{\left(\underline{Y}_k - \mathbf{H}_k\underline{\hat{X}}_{k|k-1}\right)\underline{X}_k^T\right\}\mathbf{F}_k^T.$$

We now assume that the given equality is true for $t = k$. From this and the definition of $\mathbf{K}_k^a$, we then have

$$E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)\underline{X}_{k+1}^T\right\} = \mathbf{\Sigma}_{k|k-1}^a\left[\mathbf{I} - \mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{\Sigma}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}E\left\{\left(\underline{Y}_k - \mathbf{H}_k\underline{\hat{X}}_{k|k-1}\right)\underline{X}_k^T\right\}\right]\mathbf{F}_k^T$$

$$= \mathbf{\Sigma}_{k|k-1}^a\left[\mathbf{I} - \mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{\Sigma}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}\mathbf{H}_k E\left\{\left(\underline{X}_k - \underline{\hat{X}}_{k|k-1}\right)\underline{X}_k^T\right\}\right]\mathbf{F}_k^T$$

$$= \mathbf{\Sigma}_{k|k-1}^a\left[\mathbf{I} - \mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{\Sigma}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}\mathbf{H}_k\mathbf{\Sigma}_{k|k-1}\right]\mathbf{F}_k^T = \mathbf{\Sigma}_{k|k-1}^a\left[\mathbf{I} - \mathbf{H}_k^T\mathbf{K}_k^T\right]\mathbf{F}_k^T,$$

where the last equality follows by definiton of $\mathbf{K}_k$. Applying the recursion for $\mathbf{\Sigma}_{k+1|k}^a$, we have

$$E\left\{\left(\underline{X}_j - \underline{\hat{X}}_{j|k}\right)\underline{X}_{k+1}^T\right\} = \mathbf{\Sigma}_{k+1|k}^a,$$

which shows that the given equation for $t = k+1$. The induction principle thus gives the desired result.

b. As with the measurement-update derivation in the Kalman-Bucy filter, we note here that $\underline{X}_j$ and $\underline{Y}_t$ are jointly Gaussian conditioned on $\underline{Y}_0^{t-1}$. Thus, $E\left\{\underline{X}_j|\underline{Y}_0^t\right\}$ will be as given by the recursion if we have

$$Cov\left(\underline{X}_j, \underline{Y}_t|\underline{Y}_0^{t-1}\right)\left[Cov\left(\underline{Y}_t|\underline{Y}_0^{t-1}\right)\right]^{-1} = \mathbf{K}_t^a.$$

But, since $\underline{Y}_t = \mathbf{H}_t\underline{X}_t + \underline{V}_t$, the result from Part a. implies this equality.

Exercise 9:

This is the Kalman-Bucy problem with all dimensions equal to unity, $\mathbf{F}_k \equiv 1, \mathbf{G}_k \equiv \mathbf{Q}_k \equiv 0, \mathbf{H}_k = s_k, \mathbf{R}_k \equiv \sigma^2, \underline{m}_0 \equiv \mu, \mathbf{\Sigma}_0 \equiv v^2, \underline{\hat{X}}_{n+1|n} \equiv \underline{\hat{X}}_{n|n} \equiv \hat{\theta}_n$, and $\mathbf{\Sigma}_{n|n} \equiv \mathbf{\Sigma}_{n|n} \equiv E\left\{\left(\hat{\theta}_n - \Theta\right)^2\right\}$. The desired recursions thus follow by eliminating either set of updates from (V.B.14) - (V.B.16). The resulting estimate is the same as that found in Example IV.B.2.