An Introduction to Signal Detection and Estimation - Second Edition
Chapter IV: Selected Solutions

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Exercise 1:

a.

$$\hat{\theta}_{MAP}(y) = \arg\left[\max_{1\leq\theta\leq e} \log\theta - \theta|y| - \log\theta\right] = 1.$$

b.

$$\hat{\theta}_{MMSE}(y) = \frac{\int_1^e \theta e^{-\theta|y|} d\theta}{\int_1^e e^{-\theta|y|} d\theta} = \frac{1}{|y|} + \frac{e^{-|y|} - e^{1-e|y|}}{e^{-|y|} - e^{-e|y|}}$$

Exercise 3:

$$w(\theta|y) = \frac{\theta^y e^{-\theta} e^{-\alpha\theta}}{\int_0^\infty \theta^y e^{-\theta} e^{-\alpha\theta} d\theta} = \frac{\theta^y e^{-(\alpha+1)\theta}(1+\alpha)^{y+1}}{y!}.$$

So:

$$\hat{\theta}_{MMSE}(y) = \frac{1}{y!}\int_0^\infty \theta^{y+1} e^{-(\alpha+1)\theta} d\theta (1+\alpha)^{y+1} = \frac{y+1}{\alpha+1};$$

$$\hat{\theta}_{MAP}(y) = \arg\left[\max_{\theta>0} y\log\theta - (\alpha+1)\theta\right] = \frac{y}{\alpha+1};$$

and $\hat{\theta}_{ABS}(y)$ solves

$$\int_0^{\hat{\theta}_{ABS}(y)} w(\theta|y) d\theta = \frac{1}{2}$$

which reduces to

$$\sum_{k=0}^{y} \frac{\left[\hat{\theta}_{ABS}(y)\right]^k}{k!} = \frac{1}{2} e^{\hat{\theta}_{ABS}(y)}.$$

Note that the series on the left-hand side is the truncated power series expansion of $\exp\{\hat{\theta}_{ABS}(y)\}$ so that the $\hat{\theta}_{ABS}(y)$ is the value at which this truncated series equals half of its untruncated value when the truncation point is y.

Exercise 7:

We have

$$p_\theta(y) = \begin{cases} e^{-y+\theta} & \text{if } y \geq \theta \\ 0 & \text{if } y < \theta \end{cases};$$

and

$$w(\theta) = \begin{cases} 1 & \text{if } \theta \in (0,1) \\ 0 & \text{if } \theta \notin (0,1) \end{cases}.$$

Thus,

$$w(\theta|y) = \frac{e^{-y+\theta}}{\int_0^{\min\{1,y\}} e^{-y+\theta} d\theta} = \frac{e^\theta}{e^{\min\{1,y\}} - 1},$$

for $0 \leq \theta \leq \min\{1, y\}$, and $w(\theta|y) = 0$ otherwise. This implies:

$$\hat{\theta}_{MMSE}(y) = \frac{\int_0^{\min\{1,y\}} \theta e^\theta d\theta}{e^{\min\{1,y\}} - 1} = \frac{[\min\{1,y\} - 1]e^{\min\{1,y\}} + 1}{e^{\min\{1,y\}} - 1};$$

$$\hat{\theta}_{MAP}(y) = \arg\left[\max_{0 \leq \theta \leq \min\{1,y\}} e^\theta\right] = \min\{1, y\};$$

and

$$\int_0^{\hat{\theta}_{ABS}(y)} e^\theta d\theta = \frac{1}{2}\left[e^{\min\{1,y\}} - 1\right]$$

which has the solution

$$\hat{\theta}_{ABS}(y) = \log\left[\frac{e^{\min\{1,y\}} + 1}{2}\right].$$

Exercise 8:

a. We have

$$w(\theta|y) = \frac{e^{-y+\theta}e^{-\theta}}{e^{-y}\int_0^y d\theta} = \frac{1}{y}, 0 \leq \theta \leq y,$$

and $w(\theta|y) = 0$ otherwise. That is, given y, Θ is uniformly distributed on the interval $[0, y]$. From this we have immediately that $\hat{\theta}_{MMSE}(y) = \hat{\theta}_{ABS}(y) = \frac{y}{2}$.

b. We have

$$MMSE = E\left\{Var(\Theta|Y)\right\}.$$

Since $w(\theta|y)$ is uniform on $[0, y], Var(\Theta|Y) = \frac{Y^2}{12}$. We have

$$p(y) = e^{-y}\int_0^y d\theta = ye^{-y}, y > 0,$$

from which

$$MMSE = \frac{E\{Y^2\}}{12} = \frac{1}{12}\int_0^\infty y^3 e^{-y}|dy = \frac{3!}{12} = \frac{1}{2}.$$

c. In this case,

$$p_\theta(y) = \prod_{k=1}^{n} e^{-y_k+\theta}, if 0 < \theta < \min\{y_1, \ldots, y_n\}.$$

from which

$$\hat{\theta}_{MAP}(y) = \arg\left[\max_{0<\theta<\min\{y_1,\ldots,y_n\}} \exp\left\{\sum_{k=1}^{n} y_k + (n-1)\theta\right\}\right] = \min\{y_1, \ldots, y_n\}.$$

Exercise 13:

a. We have

$$p_\theta(\underline{y}) = \theta^{T(\underline{y})}(1-\theta)^{(n-T(\underline{y}))},$$

where

$$T(\underline{y}) = \sum_{k=1}^{n} y_k.$$

Rewriting this as

$$p_\theta(\underline{y}) = C(\phi)e^{\phi T(\underline{y})}$$

with $\phi = \log(\theta/(1-\theta))$, and $C(\phi) = e^{n\phi}$ we see from the Completeness Theorem for Exponential Families that $T(\underline{y})$ is a complete sufficient statistic for ϕ and hence for θ. (Assuming θ ranges throughout $(0,1)$.) Thus, any unbiased function of T is an MVUE for θ. Since $E_\theta\{T(\underline{Y})\} = n\theta$, such an estimate is given by

$$\hat{\theta}_{MV}(\underline{y}) = \frac{T(\underline{y})}{n} = \frac{1}{n}\sum_{k=1}^{n} y_k.$$

b.

$$\hat{\theta}_{ML}(\underline{y}) = \arg\left\{\max_{0<\theta<1} \theta^{T(\underline{y})}(1-\theta)^{(n-T(\underline{y}))}\right\} = T(\underline{y})/n = \hat{\theta}_{MV}(\underline{y}).$$

Since the MLE equals the MVUE, we have immediately that $E_\theta\{\hat{\theta}_{ML}(\underline{Y})\} = \theta$. The variance of $\hat{\theta}_{ML}(\underline{Y})$ is easily computed to be $\theta(1-\theta)/n$.

c. We have

$$\frac{\partial^2}{\partial\theta^2}\log p_\theta(\underline{Y}) = -\frac{T(\underline{Y})}{\theta^2} - \frac{n-T(\underline{Y})}{(1-\theta)^2},$$

from which

$$I_\theta = \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n}{\theta(1-\theta)}.$$

The CRLB is thus

$$CRLB = \frac{1}{I_\theta} = \frac{\theta(1-\theta)}{n} = Var_\theta(\hat{\theta}_{ML}(\underline{Y})).$$

Exercise 15:

We have

$$p_\theta(y) = \frac{e^{-\theta}\theta^y}{y!}, \quad y \in 0, 1, \ldots.$$

Thus

$$\frac{\partial}{\partial\theta} \log p_\theta(y) = \frac{\partial}{\partial\theta}(-\theta + y\log\theta) = -1 + \frac{y}{\theta},$$

and

$$\frac{\partial^2}{\partial\theta^2} \log p_\theta(y) = -\frac{y}{\theta^2} < 0.$$

So

$$\hat{\theta}_{ML}(y) = y.$$

Since Y is Poisson, we have $E_\theta\{\hat{\theta}_{ML}(Y)\} = Var_\theta(\hat{\theta}_{ML}(Y)) = \theta$. So, $\hat{\theta}_{ML}$ is unbiased.

Fisher's information is given by

$$I_\theta = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2} \log p_\theta(Y)\right\} = \frac{E_\theta\{Y\}}{\theta^2} = \frac{1}{\theta}.$$

So the CRLB is θ, which equals $Var_\theta(\hat{\theta}_{ML}(Y))$. (Hence, the MLE is an MVUE in this case.)

Exercise 20:

a. Note that $Y_1, Y_2, \ldots, Y_n$, are independent with Y_k having the $\mathcal{N}(0, 1+\theta s_k^2)$ distribution. Thus,

$$\frac{\partial}{\partial\theta} \log p_\theta(\underline{y}) = \sum_{k=1}^{n} \frac{\partial}{\theta}\left\{-\frac{1}{2}\log(1+\theta s_k^2) - \frac{y_k^2}{2(1+\theta s_k^2)}\right\}$$

$$= -\frac{1}{2}\sum_{k=1}^{n}\left\{\frac{s_k^2}{1+\theta s_k^2} - \frac{y_k^2 s_k^2}{(1+\theta s_k^2)^2}\right\},$$

from which the likelihood equation becomes

$$\sum_{k=1}^{n} \frac{s_k^2(y_k^2 - 1 - \hat{\theta}_{ML}(\underline{y})s_k^2)}{(1+\hat{\theta}_{ML}(\underline{y})s_k^2)^2} = 0.$$

b.

$$I_\theta = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2} \log p_\theta(\underline{Y})\right\} = \sum_{k=1}^{n}\left\{\frac{s_k^4 E_\theta\{Y_k^2\}}{(1+\theta s_k^2)^3} - \frac{s_k^4}{2(1+\theta_k^2)^2}\right\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\frac{s_k^4}{(1+\theta s_k^2)^2}.$$

So the CRLB is

$$\frac{2}{\sum_{k=1}^{n} \frac{s_k^4}{(1+\theta s_k^2)^2}}.$$

c. With $s_k^2 = 1$, the likelihood equation yields the solution

$$\hat{\theta}_{ML}(\underline{y}) = \left(\frac{1}{n} \sum_{k=1}^{n} y_k^2 \right) - 1,$$

which is seen to yield a maximum of the likelihood function.

d. We have

$$E_\theta \left\{ \hat{\theta}_{ML}(\underline{Y}) \right\} = \left(\frac{1}{n} \sum_{k=1}^{n} E_\theta \left\{ Y_k^2 \right\} \right) - 1 = \theta.$$

Similarly, since the $Y_k's$ are independent,

$$Var_\theta \left(\hat{\theta}_{ML}(\underline{Y} \right) = \frac{1}{n^2} \sum_{k=1}^{n} Var_\theta \left(Y_k^2 \right) = \frac{1}{n^2} \sum_{k=1}^{n} 2(! + \theta)^2 = \frac{2(1+\theta)^2}{n}.$$

Thus, the bias of the MLE is 0 and the variance of the MLE equals the CRLB. (Hence, the MLE is an MVUE in this case.)

Exercise 22:

a. Note that

$$p_\theta(\underline{y}) = exp \left\{ \sum_{k=1}^{n} \log F(y_k)/\theta \right\},$$

which impies that the statistic $\sum_{k=1}^{n} \log F(y_k)$ is a complete sufficient statistic for θ via the Completeness Theorem for Exponential Families. We have

$$E_\theta \left\{ \sum_{k=1}^{n} \log F(Y_k) \right\} = nE_\theta\{\log F(Y_1)\} = \frac{n}{\theta} \int_{-\infty}^{\infty} \log F(Y_1) \left[F(y_1)\right]^{(1-\theta)/\theta} f(y_1) dy_1.$$

Noting that $d \log F(y_1) = \frac{f(y_1)}{F(y_1)} dy_1$, and that $\left[F(y_1)\right]^{1/\theta} = exp\{\log F(y_1)/\theta\}$, we can make the substitution $x = \log F(y_1)$ to yield

$$E_\theta \left\{ \sum_{k=1}^{n} \log F(Y_k) \right\} = \frac{n}{\theta} \int_{-\infty}^{0} x e^{x/\theta} dx = -n\theta.$$

Thus, we have

$$E_\theta \left\{ \hat{\theta}_{MV}(\underline{Y}) \right\} = \theta,$$

which implies that $\hat{\theta}_{MV}$ is an MVUE since it is an unbiased function of a complete sufficient statistic.

b. [Correction: Note that, for the given prior, the prior mean should be $E\{\Theta\} = \frac{c}{m-1}$.]

It is straightforward to see that $w(\theta|\underline{y})$ is of the same form as the prior with c replaced by $c - \sum_{k=1}^{n} \log F(y_k)$, and m replaced by $n + m$. Thus, by inspection

$$E\{\Theta|Y\} = \frac{c - \sum_{k=1}^{n} \log F(Y_k)}{m + n - 1},$$

which was to be shown. [Again, the necessary correction has been made.]

c. In this example, the prior and posterior distributions have the same form. The only change is that the parameters of that distribution are updated as new data is observed. A prior with this property is said to be a *reproducing prior*. The prior parameters , c and m, can be thought of as coming from an earlier sample of size m. As n becomes large compared to m, the importance of these prior parameters in the estimate diminishes. Note that $\sum_{k=1}^{n} \log F(Y_k)$ behaves like $nE\{\log F(Y_1)\}$ for large n. Thus, with $n \gg m$, the estimate is appropximately given by the MVUE of Part a. Altenatively, with $m \gg n$, the estimate is approximately the prior mean, $c/(m-1)$. Between these two extremes, there is a balance between prior and observed information.

Exercise 23:

a. The log-likelihood function is

$$\log p(\underline{y}|A, \phi) = -\frac{1}{2\sigma^2} \sum_{k=1}^{n} \left[y_k - A \sin\left(\frac{k\pi}{2} + \phi\right)\right]^2 - \frac{n}{2} \log(2\pi\sigma^2).$$

The likelihood equations are thus:

$$\sum_{k=1}^{n} \left[y_k - \hat{A} \sin\left(\frac{k\pi}{2} + \hat{\phi}\right)\right] \sin\left(\frac{k\pi}{2} + \hat{\phi}\right) = 0,$$

and

$$\hat{A} \sum_{k=1}^{n} \left[y_k - \hat{A} \sin\left(\frac{k\pi}{2} + \hat{\phi}\right)\right] \cos\left(\frac{k\pi}{2} + \hat{\phi}\right) = 0.$$

These equations are solved by the estimates:

$$\hat{A}_{ML} = \sqrt{y_c^2 + y_s^2},$$

$$\hat{\phi}_{ML} = \tan^{-1}\left(\frac{y_c}{y_s}\right),$$

where

$$y_c = \frac{1}{n} \sum_{k=1}^{n} y_k \cos\left(\frac{k\pi}{2}\right) \equiv \frac{1}{n} \sum_{k=1}^{n/2} (-1)^k y_{2k},$$

and

$$y_s = \frac{1}{n}\sum_{k=1}^{n} y_k \sin\left(\frac{k\pi}{2}\right) \equiv \frac{1}{n}\sum_{k=1}^{n/2}(-1)^{k+1}y_{2k-1}.$$

b. Appending the prior to the above problem yields MAP estimates:

$$\hat{\phi}_{MAP} = \hat{\phi}_{ML},$$

and

$$\hat{A}_{MAP} = \frac{\hat{A}_{ML} + \sqrt{\left(\frac{r}{n}\right)^2 + \frac{2(1+\alpha)\sigma^2}{n}}}{1+\alpha},$$

where $\alpha \equiv \frac{2\sigma^2}{n\beta^2}$.

c. Note that, when $\beta \to \infty$ (and the prior "diffuses"), the MAP estimate of A does not approach the MLE of A. However, as $n \to \infty$, the MAP estimate does approach the MLE.