We see from Table 12.6 that, in general, for a given rate R_{tb} and constraint length ν, the minimum distance d_{min} of the best block (tail-biting convolutional) code increases, or the number of nearest neighbors decreases, as the information block length K^* increases. Once K^* reaches a certain value, though, the minimum distance d_{min} of the best block (tail-biting convolutional) code is limited by the free distance d_{free} of the best terminated convolutional code with constraint length ν, and no further increase in d_{min} is possible; however, the number of nearest neighbors $A_{d_{min}}$ continues to grow linearly with K^*. Once this limit is reached, the generator sequences $\mathbf{g}^{(j)}$ (parity-check sequences $\mathbf{h}^{(j)}$ in the rate $R_{tb} = 2/3$ case) and the minimum distance d_{min} stay the same, and in Table 12.6 we simply list the growth rate of $A_{d_{min}}$. In other words, for a given R_{tb} and ν, block (tail-biting convolutional) codes improve as K^* increases up to a point, and then the codes get worse. Similarly, we can see from Table 12.6 that for a given R_{tb} and K^*, block (tail-biting convolutional) codes improve as ν increases up to a point, and then d_{min} and $A_{d_{min}}$ remain the same. Thus, the best block (tail-biting convolutional) codes are obtained by choosing the length K^* or the constraint length ν only as large as is needed to achieve the desired combination of d_{min} and $A_{d_{min}}$. It is worth noting that many of the best binary block codes can be represented as tail-biting convolutional codes, and thus they can be decoded using the ML (Viterbi) or MAP (BCJR) soft-decision decoding algorithms (see Problem 12.39).

PROBLEMS

12.1 Draw the trellis diagram for the (3, 2, 2) code listed in Table 12.1(d) and an information sequence of length $h = 3$ blocks. Find the codeword corresponding to the information sequence $\mathbf{u} = (11, 01, 10)$. Compare the result with (11.16) in Example 11.2.

12.2 Show that the path $\mathbf{v}$ that maximizes $\sum_{l=0}^{N-1} \log P(r_l|v_l)$ also maximizes $\sum_{l=0}^{N-1} c_2[\log P(r_l|v_l) + c_1]$, where c_1 is any real number and c_2 is any positive real number.

12.3 Find the integer metric table for the DMC of Figure 12.3 when $c_1 = 1$ and $c_2 = 10$. Use the Viterbi algorithm to decode the received sequence $\mathbf{r}$ of Example 12.1 with this integer metric table and the trellis diagram of Figure 12.1. Compare your answer with the result of Example 12.1.

✱ **12.4** Consider a binary-input, 8-ary output DMC with transition probabilities $P(r_l|v_l)$ given by the following table:

$v_l^{(j)}$ \ $r_l^{(j)}$	0_1	0_2	0_3	0_4	1_4	1_3	1_2	1_1
0	0.434	0.197	0.167	0.111	0.058	0.023	0.008	0.002
1	0.002	0.008	0.023	0.058	0.111	0.167	0.197	0.434

Find the metric table and an integer metric table for this channel.

✱ **12.5** Consider the (2, 1, 3) encoder of Figure 11.1 with

$$\mathbf{G}(D) = [1 + D^2 + D^3 \quad 1 + D + D^2 + D^3]$$

a. Draw the trellis diagram for an information sequence of length $h = 4$.

b. Assume a codeword is transmitted over the DMC of Problem 12.4. Use the Viterbi algorithm to decode the received sequence $\mathbf{r} = (1_2 1_1, 1_2 0_1, 0_3 0_1, 0_1 1_3, 1_2 0_2, 0_3 1_1, 0_3 0_2)$.

12.6 The DMC of Problem 12.4 is converted to a BSC by combining the soft-decision outputs $0_1, 0_2, 0_3$, and 0_4 into a single hard-decision output 0, and the soft-decision outputs $1_1, 1_2, 1_3$, and 1_4 into a single hard-decision output 1. A codeword from the code of Problem 12.5 is transmitted over this channel. Use the Viterbi algorithm to decode the hard-decision version of the received sequence in Problem 12.5 and compare the result with Problem 12.5.

12.7 A codeword from the code of Problem 12.5 is transmitted over a continuous-output AWGN channel. Use the Viterbi algorithm to decode the (normalized by $\sqrt{E_S}$) received sequence $\mathbf{r} = (+1.72, +0.93, +2.34, -3.42, -0.14, -2.84, -1.92, +0.23, +0.78, -0.63, -0.05, +2.95, -0.11, -0.55)$.

12.8 Consider a binary-input, continuous-output AWGN channel with signal-to-noise ratio $E_s/N_0 = 0$ dB.

a. Sketch the conditional pdf's of the (normalized by $\sqrt{E_s}$) received signal r_l given the transmitted bits $v_l = \pm 1$.

b. Convert this channel into a binary-input, 4-ary output symmetric DMC by placing quantization thresholds at the values $r_l = -1, 0$, and $+1$, and compute the transition probabilities for the resulting DMC.

c. Find the metric table and an integer metric table for this DMC.

d. Repeat parts (b) and (c) using quantization thresholds $r_l = -2, 0$, and $+2$.

12.9 Show that (12.21) is an upper bound on P_d for d even.

12.10 Consider the (2, 1, 3) encoder of Problem 12.5. Evaluate the upper bounds on event-error probability (12.25) and bit-error probability (12.29) for a BSC with transition probability

a. $p = 0.1$,

b. $p = 0.01$.

(*Hint*: Use the WEFs derived for this encoder in Example 11.12.)

12.11 Repeat Problem 12.10 using the approximate expressions for $P(E)$ and $P_b(E)$ given by (12.26) and (12.30).

12.12 Consider the (3, 1, 2) encoder of (12.1). Plot the approximate expression (12.36) for bit-error probability $P_b(E)$ on a BSC as a function of E_b/N_0 in decibels. Also plot on the same set of axes the approximate expression (12.37) for $P_b(E)$ without coding. The *coding gain* (in decibels) is defined as the difference between the E_b/N_0 ratio needed to achieve a given bit-error probability with coding and without coding. Plot the coding gain as a function of $P_b(E)$. Find the value of E_b/N_0 for which the coding gain is 0 dB, that is, the *coding threshold*.

12.13 Repeat Problem 12.12 for an AWGN channel with unquantized demodulator outputs, that is, a continuous-output AWGN channel, using the approximate expression for $P_b(E)$ given in (12.46).

12.14 Consider using the (3, 1, 2) encoder of (12.1) on the DMC of Problem 12.4. Calculate an approximate value for the bit-error probability $P_b(E)$ based on the bound of (12.39b). Now, convert the DMC to a BSC, as described in Problem 12.6; compute an approximate value for $P_b(E)$ on this BSC using (12.30); and compare the two results.

12.15 Prove that the rate $R = 1/2$ quick-look-in encoders defined by (12.58) are noncatastrophic.

12.16 Consider the following two nonsystematic feedforward encoders: (1) the encoder for the (2, 1, 7) optimum code listed in Table 12.1(c) and (2) the encoder for the (2, 1, 7) quick-look-in code listed in Table 12.2. For each of these codes find
   a. the soft-decision asymptotic coding gain γ;
   b. the approximate event-error probability on a BSC with $p = 10^{-2}$;
   c. the approximate bit-error probability on a BSC with $p = 10^{-2}$;
   d. the error probability amplification factor A.

12.17 Using trial-and-error methods, construct a (2, 1, 7) systematic feedforward encoder with maximum d_{free}. Repeat Problem 12.16 for this code.

12.18 Consider the (15,7) and (31,16) cyclic BCH codes. For each of these codes find
   a. the polynomial generator matrix and a lower bound on d_{free} for the rate $R = 1/2$ convolutional code derived from the cyclic code using Construction 12.1;
   b. the polynomial generator matrix and a lower bound on d_{free} for the rate $R = 1/4$ convolutional code derived from the cyclic code using Construction 12.2.

   (*Hint:* d_h is at least one more than the maximum number of consecutive powers of α that are roots of $\mathbf{h}(X)$.)

12.19 Consider the (2, 1, 1) systematic feedforward encoder with $\mathbf{G}(D) = [1 \quad 1 + D]$.
   a. For a continuous-output AWGN channel and a truncated Viterbi decoder with path memory $\tau = 2$, decode the received sequence $\mathbf{r} =$ (+1.5339, +0.6390, −0.6747, −3.0183, +1.5096, +0.7664, −0.4019, +0.3185, +2.7121, −0.7304, +1.4169, −2.0341, +0.8971, −0.3951, +1.6254, −1.1768, +2.6954, −1.0575) corresponding to an information sequence of length $h = 8$. Assume that at each level the survivor with the best metric is selected and that the information bit τ time units back on this path is decoded.
   b. Repeat (a) for a truncated Viterbi decoder with path memory $\tau = 4$.
   c. Repeat (a) for a Viterbi decoder without truncation.
   d. Are the final decoded paths the same in all cases ? Explain.

12.20 Consider the (3, 1, 2) encoder of Problem 11.19.
   a. Find $A_1(W, X, L)$, $A_2(W, X, L)$, and $A_3(W, X, L)$.
   b. Find τ_{min}.
   c. Find $d(\tau)$ and $A_{d(\tau)}$ for $\tau = 0, 1, 2, \cdots, \tau_{min}$.
   d. Find an expression for $\lim_{\tau \to \infty} d(\tau)$.

12.21 A codeword from the trellis diagram of Figure 12.1 is transmitted over a BSC. To determine correct symbol synchronization, each of the three 21-bit subsequences of the sequence

$$\mathbf{r} = 0\,1\,1\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,0\,0\,0\,1\,1\,1$$

must be decoded, where the two extra bits in $\mathbf{r}$ are assumed to be part of a preceding and/or a succeeding codeword. Decode each of these subsequences and determine which one is most likely to be the correctly synchronized received sequence.

12.22 Consider the binary-input, continuous-output AWGN channel of Problem 12.8.
   a. Using the optimality condition of (12.84), calculate quantization thresholds for DMCs with $Q = 2, 4$, and 8 output symbols. Compare the thresholds obtained for $Q = 4$ with the values used in Problem 12.8.
   b. Find the value of the Bhattacharyya parameter D_0 for each of these channels and for a continuous-output AWGN channel.
   c. Fixing the signal energy $\sqrt{E_s} = 1$ and allowing the channel SNR E_s/N_0 to vary, determine the increase in the SNR required for each of the DMCs to achieve

the same value of D_0 as the continuous-output channel. This SNR difference is called the *decibel loss* associated with receiver quantization. (*Note:* Changing the SNR also changes the quantization thresholds.)

(*Hint:* You will need to write a computer program to solve this problem.)

12.23 Verify that the two expressions given in (12.89) for the modified metric used in the SOVA algorithm are equivalent.

12.24 Define $L(r) = \ln \lambda(r)$ as the log-likelihood ratio, or L-value, of a received symbol r at the output of an unquantized binary input channel. Show that the L-value of an AWGN channel with binary inputs $\pm\sqrt{E_s}$ and SNR E_s/N_0 is given by

$$L(r) = (4\sqrt{E_s}/N_0)r.$$

12.25 Verify that the expressions given in (12.98) are correct, and find the constant c.

12.26 Consider the encoder, channel, and received sequence of Problem 12.19.

a. Use the SOVA with full path memory to produce a soft output value for each decoded information bit.

b. Repeat (a) for the SOVA with path memory $\tau = 4$.

12.27 Derive the expression for the backward metric given in (12.117).

12.28 Verify the derivation of (12.123) and show that $A_l = e^{-\frac{L_a(u_l)/2}{1+e^{-L_a(u_l)}}}$ is independent of the actual value of u_l.

12.29 Derive the expressions for the max*(x, y) and max*(x, y, z) functions given in (12.127) and (12.131), respectively.

12.30 Consider the encoder and received sequence of Problem 12.19.

a. For an AWGN channel with $E_s/N_0 = 1/2$ (−3 dB), use the log-MAP version of the BCJR algorithm to produce a soft output value for each decoded information bit. Find the decoded information sequence $\hat{\mathbf{u}}$.

b. Repeat (a) using the Max-log-MAP algorithm.

12.31 Repeat Problem 12.5 using the probability-domain version of the BCJR algorithm.

12.32 Show that using the normalized forward and backward metrics $A_l(s)$ and $B_l(s')$ instead of $\alpha_l(s)$ and $\beta_l(s')$, respectively, to evaluate the joint pdf's in (12.115) has no effect on the APP L-values computed using (12.111).

12.33 Verify all the computations leading to the determination of the final APP L-values in Example 12.9.

12.34 Repeat Example 12.9 for the case when the DMC is converted to a BSC, as described in Problem 12.6, and the received sequence $\mathbf{r}$ is replaced by its hard-decision version. Compare the final APP L-values in the two cases.

12.35 Consider an 8-state rate $R = 1/2$ mother code with generator matrix

$$\mathbf{G}(D) = [1 + D + D^3 \quad 1 + D^2 + D^3].$$

Find puncturing matrices $\mathbf{P}$ for the rate $R = 2/3$ and $R = 3/4$ punctured codes that give the best free distances. Compare your results with the free distances obtained using the 8-state mother code in Table 12.4.

12.36 Prove that the subcode corresponding to any nonzero state S_i, $i \neq 0$, in a tail-biting convolutional code is a coset of the subcode corresponding to the all-zero state S_0.

12.37 For the rate $R = 1/2$ feedback encoder tail-biting trellis in Figure 12.24(b), determine the parameters d_{min} and $A_{d_{min}}$ for information block lengths $K^* = 7, 8$, and 9. Is it possible to form a tail-biting code in each of these cases?

12.38 Verify that the row space of the tail-biting generator matrix in (12.164) is identical to the tail-biting code of Table 12.5(a).