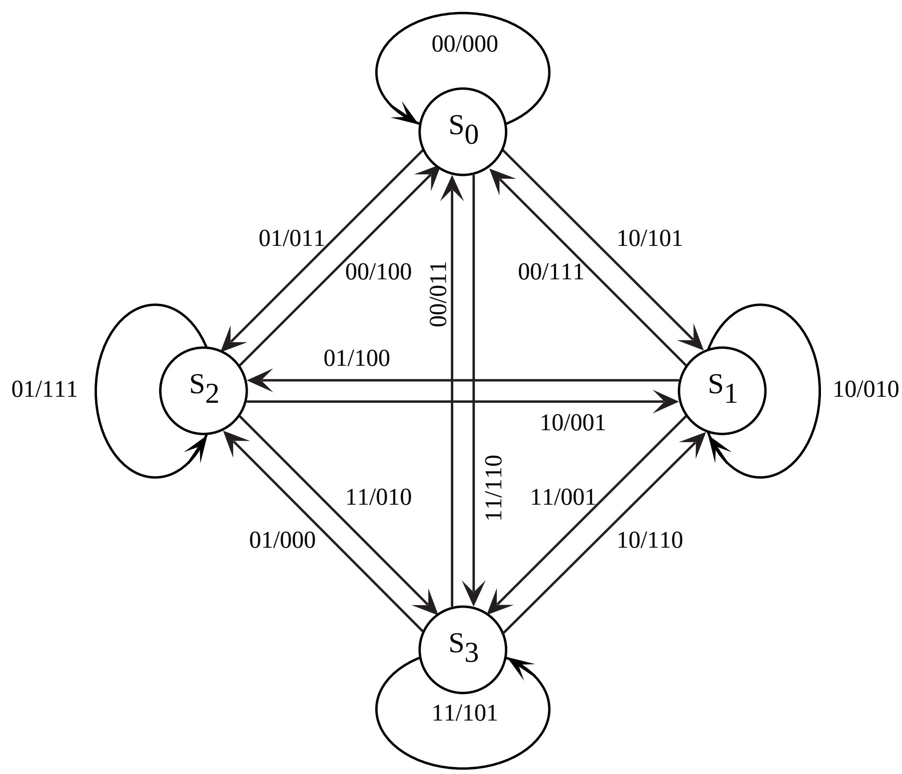
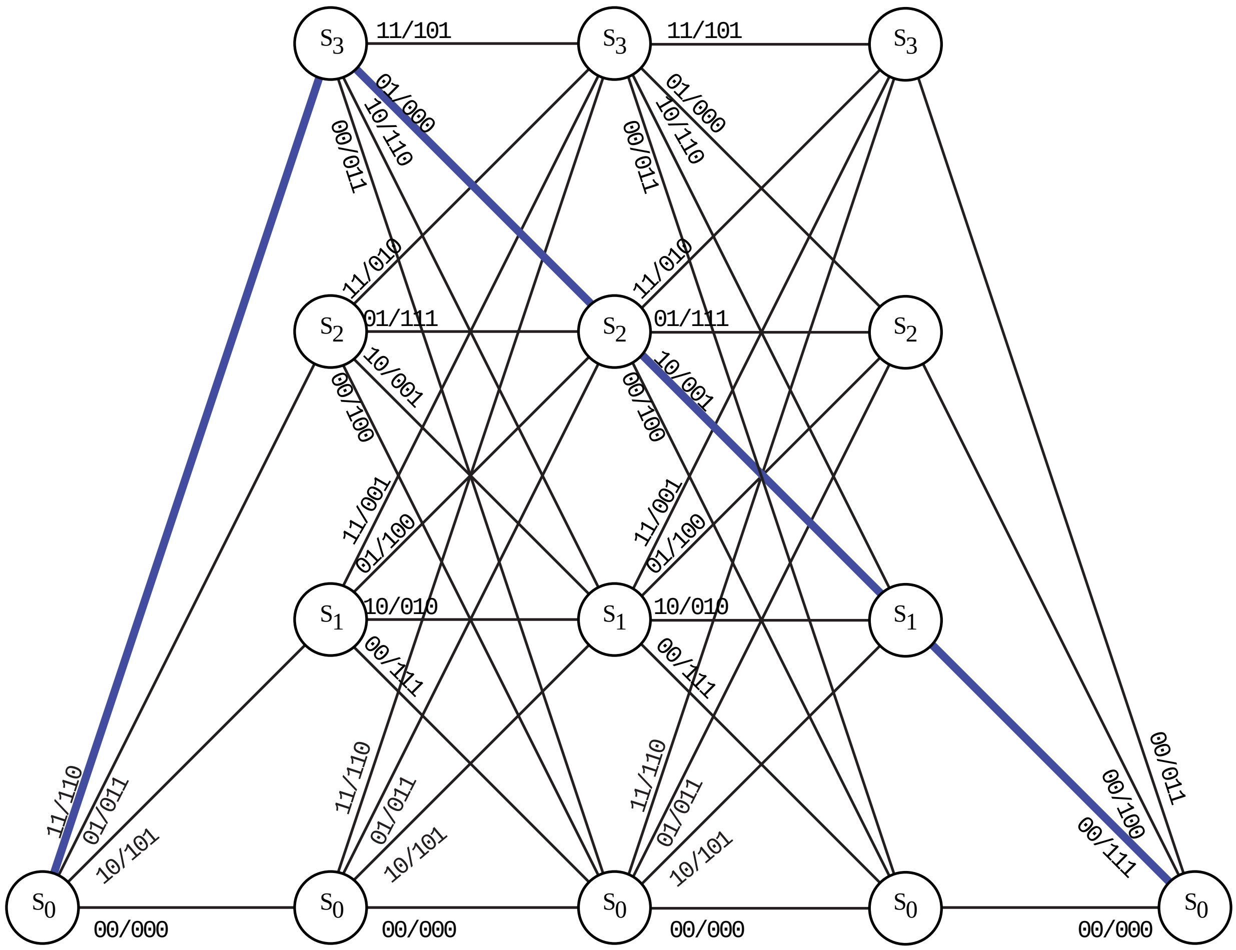
Chapter 12

Optimum Decoding of Convolutional Codes

12.1 (Note: The problem should read " for the (3,2,2) encoder in Example 11.2 "rather than " for the (3,2,2) code in Table 12.1(d)".) The state diagram of the encoder is given by:



From the state diagram, we can draw a trellis diagram containing $h + m + 1 = 3 + 1 + 1 = 5$ levels as shown below:



Hence, for $\mathbf{u} = (11, 01, 10)$,

$$\begin{aligned} \mathbf{v}^{(0)} &= (1001) \\ \mathbf{v}^{(1)} &= (1001) \\ \mathbf{v}^{(2)} &= (0011) \end{aligned}$$

and

$$\mathbf{v} = (110, 000, 001, 111),$$

agreeing with (11.16) in Example 11.2. The path through the trellis corresponding to this codeword is shown highlighted in the figure.

12.2 Note that

$$\begin{aligned}\sum_{l=0}^{N-1} c_2 \left[\log P(r_l|v_l) + c_1\right] &= \sum_{l=0}^{N-1} \left[c_2 \log P(r_l|v_l) + c_2 c_1\right] \\ &= c_2 \sum_{l=0}^{N-1} \log P(r_l|v_l) + N c_2 c_1.\end{aligned}$$

Since

$$\max_{\mathbf{v}} \left\{ c_2 \sum_{l=0}^{N-1} \log P(r_l|v_l) + N c_2 c_1 \right\} = c_2 \max_{\mathbf{v}} \left\{ \sum_{l=0}^{N-1} \log P(r_l|v_l) \right\} + N c_2 c_1$$

if C_2 is positive, any path that maximizes $\sum_{l=0}^{N-1} \log P(r_l|v_l)$ also maximizes $\sum_{l=0}^{N-1} c_2[\log P(r_l|v_l) + c_1]$.

12.3 The integer metric table becomes:

	0_1	0_2	1_2	1_1
0	6	5	3	0
1	0	3	5	6

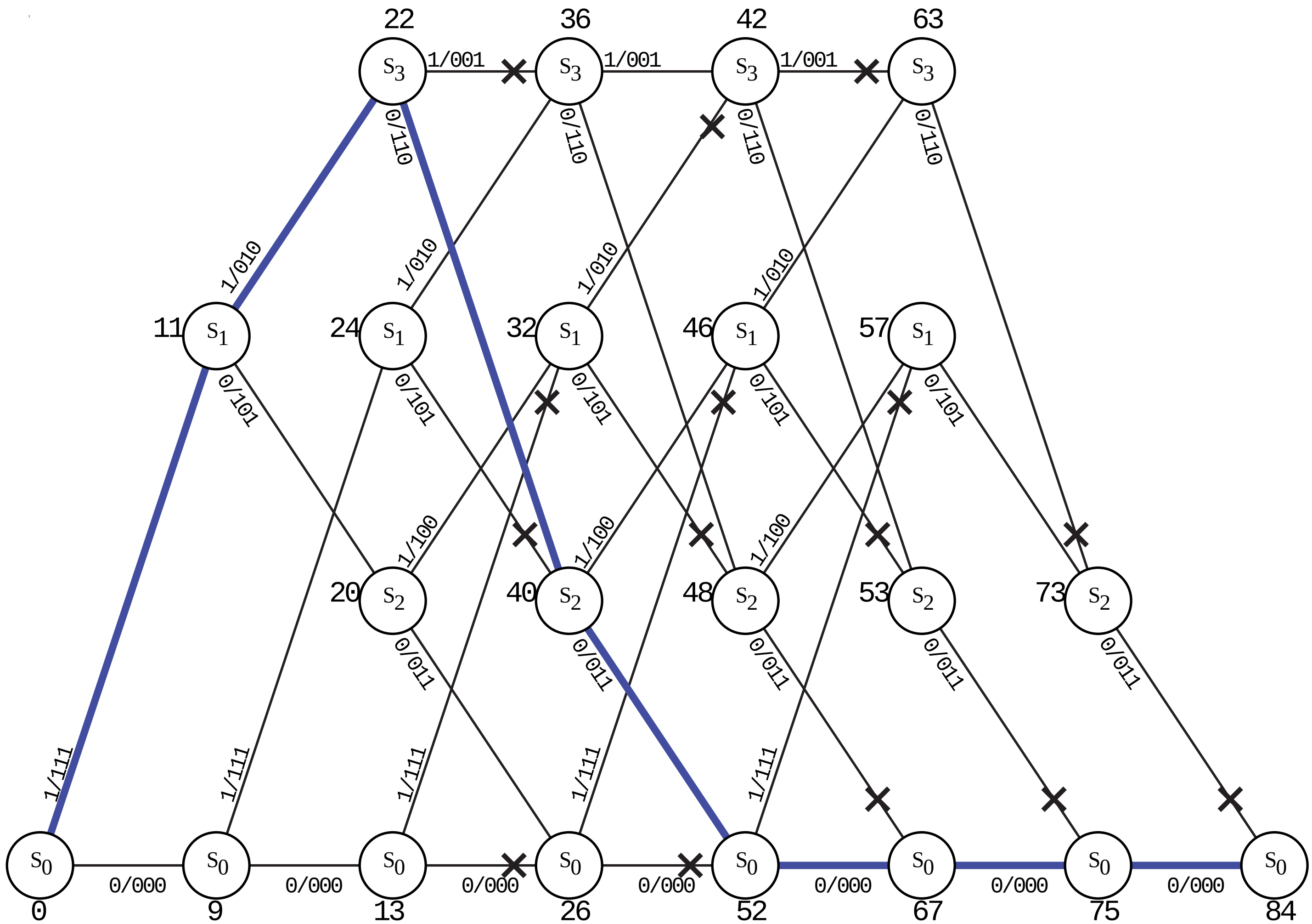
The received sequence is $\mathbf{r} = (1_1 1_2 0_1, 1_1 1_1 0_2, 1_1 1_1 0_1, 1_1 1_1 1_1, 0_1 1_2 0_1, 1_2 0_2 1_1, 1_2 0_1 1_1)$. The decoded sequence is shown in the figure below, and the final survivor is

$$\hat{\mathbf{v}} = (111, 010, 110, 011, 000, 000, 000),$$

which yields a decoded information sequence of

$$\hat{\mathbf{u}} = (11000).$$

This result agrees with Example 12.1.


22
36
42
63
S_3
S_3
S_3
S_3
1/001
1/001
1/001
0/110
0/110
0/110
0/110
1/010
1/010
1/010
1/010
11
24
32
46
57
S_1
S_1
S_1
S_1
S_1
0/101
0/101
0/101
0/101
0/101
1/100
1/100
1/100
20
40
48
53
73
S_2
S_2
S_2
S_2
S_2
0/011
0/011
0/011
0/011
0/011
1/111
1/111
1/111
1/111
1/111
S_0
S_0
S_0
S_0
S_0
S_0
S_0
S_0
0/000
0/000
0/000
0/000
0/000
0/000
0/000
0
9
13
26
52
67
75
84

12.4 For the given channel transition probabilities, the resulting metric table is:

	0_1	0_2	0_3	0_4	1_4	1_3	1_2	1_1
0	−0.363	−0.706	−0.777	−0.955	−1.237	−1.638	−2.097	−2.699
1	−2.699	−2.097	−1.638	−1.237	−0.955	−0.777	−0.706	−0.363

To construct an integer metric table, choose $c_1 = 2.699$ and $c_2 = 4.28$. Then the integer metric table becomes:

	0_1	0_2	0_3	0_4	1_4	1_3	1_2	1_1
0	10	9	8	7	6	5	3	0
1	0	3	5	6	7	8	9	10

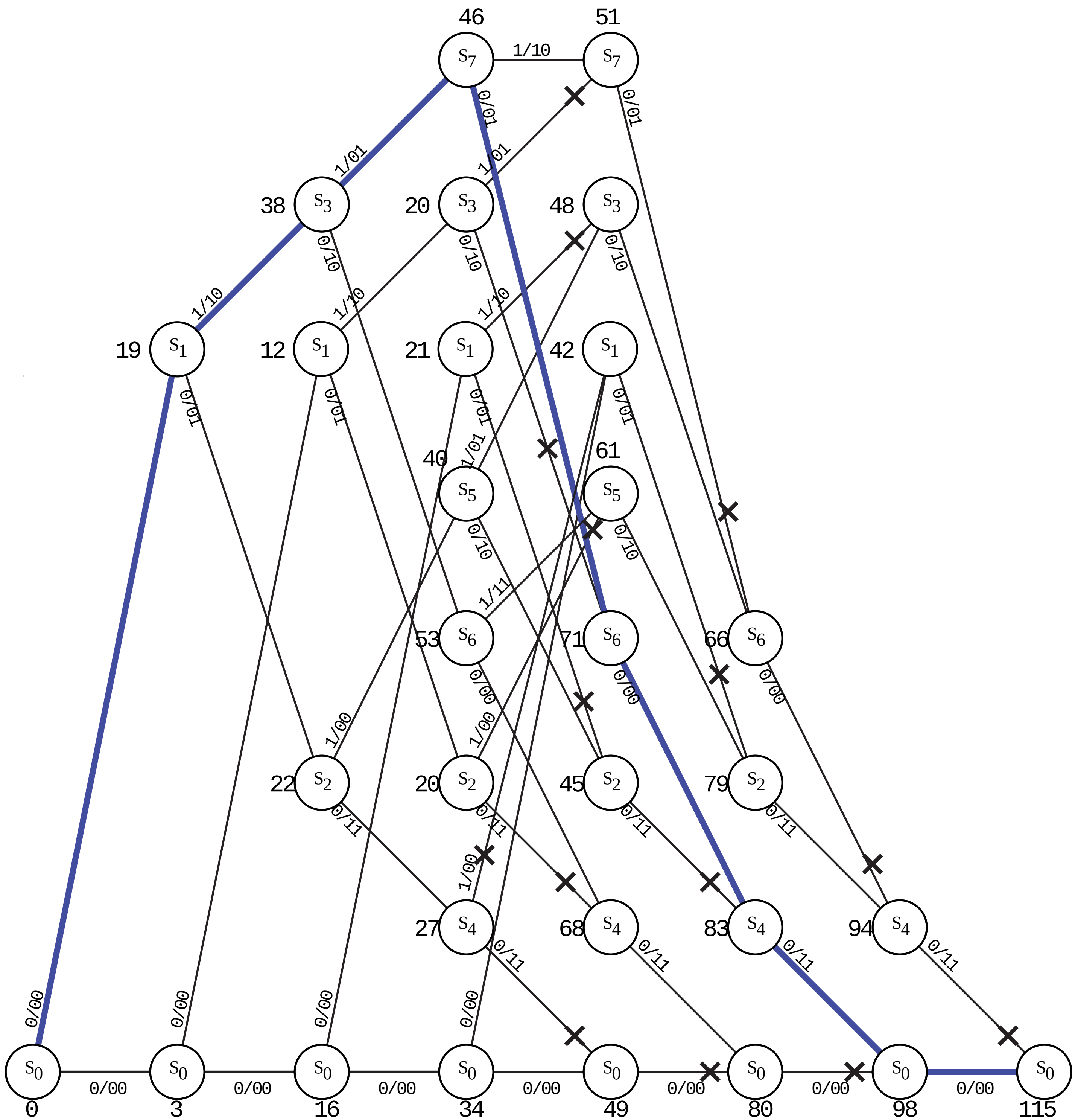
12.5 (a) Referring to the state diagram of Figure 11.13(a), the trellis diagram for an information sequence of length $h = 4$ is shown in the figure below.

(b) After Viterbi decoding the final survivor is

$$\hat{\mathbf{v}} = (11, 10, 01, 00, 11, 00).$$

This corresponds to the information sequence

$$\hat{\mathbf{u}} = (1110).$$


46
51
1/10
0/01
1/01
38
20
48
0/10
1/10
19
12
21
42
0/01
40
61
1/11
53
71
66
0/00
1/00
22
20
45
79
0/11
27
68
83
94
0/00
0
3
16
34
49
80
98
115

12.6 Combining the soft decision outputs yields the following transition probabilities:

	0	1
0	0.909	0.091
1	0.091	0.909

For hard decision decoding, the metric is simply Hamming distance. For the received sequence

$$\mathbf{r} = (11, 10, 00, 01, 10, 01, 00),$$

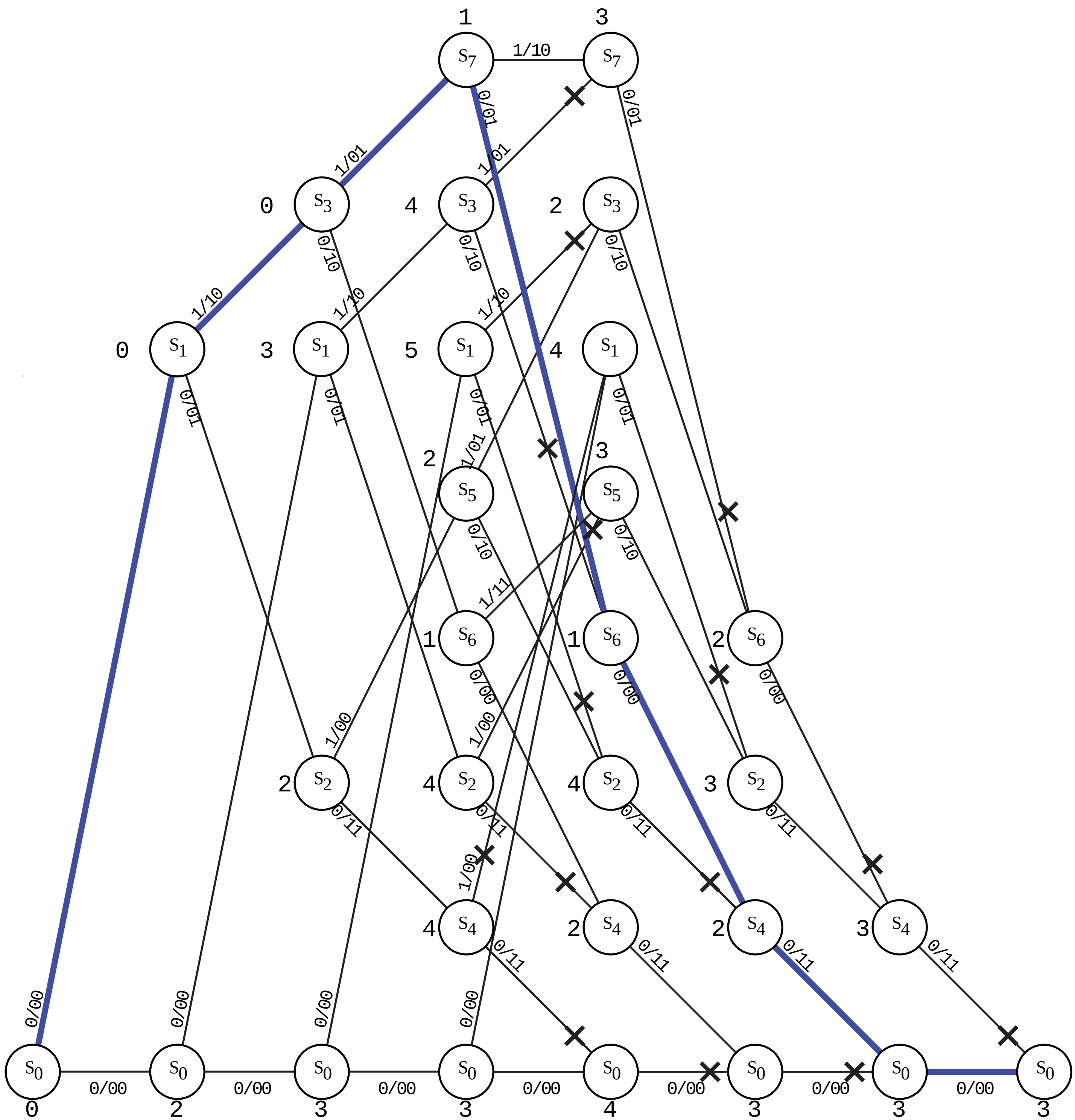
the decoding trellis is as shown in the figure below, and the final survivor is

$$\hat{\mathbf{v}} = (11, 10, 01, 01, 00, 11, 00),$$

which corresponds to the information sequence

$$\hat{\mathbf{u}} = (1110).$$

This result matches the result obtained using soft decisions in Problem 12.5.


1
3
S7
1/10
S7
0/01
0/01
1/01
1/01
0
S3
4
S3
2
S3
0/10
0/10
0/10
1/10
1/10
1/10
0
S1
3
S1
5
S1
4
S1
0/01
0/01
0/01
0/01
1/01
2
S5
3
S5
0/10
0/10
1/11
1
S6
1
S6
2
S6
0/00
0/00
0/00
1/00
1/00
2
S2
4
S2
4
S2
3
S2
0/11
0/11
0/11
0/11
1/00
4
S4
2
S4
2
S4
3
S4
0/11
0/11
0/11
0/11
0/00
0/00
0/00
0/00
S0
S0
S0
S0
S0
S0
S0
S0
0/00
0/00
0/00
0/00
0/00
0/00
0/00
0
2
3
3
4
3
3
3

12.9 **Proof:** For d even,

$$\begin{aligned}
P_d &= \frac{1}{2}\binom{d}{d/2}p^{d/2}(1-p)^{d/2} + \sum_{e=(d/2)+1}^{d}\binom{d}{e}p^e(1-p)^{d-e} \\
&< \sum_{e=(d/2)}^{d}\binom{d}{e}p^e(1-p)^{d-e} \\
&< \sum_{e=(d/2)}^{d}\binom{d}{e}p^{d/2}(1-p)^{d/2} \\
&= p^{d/2}(1-p)^{d/2}\sum_{e=(d/2)}^{d}\binom{d}{e} \\
&< 2^d p^{d/2}(1-p)^{d/2}
\end{aligned}$$

and thus (12.21) is an upper bound on P_d for d even. Q. E. D.

12.10 The event error probability is bounded by (12.25)

$$P(E) < \sum_{d=d_{free}}^{\infty} A_d P_d < A(X)\big|_{X=2\sqrt{p(1-p)}}.$$

From Example 11.12,

$$A(X) = \frac{X^6 + X^7 - X^8}{1-2X-X^3} = X^6 + 3X^7 + 5X^8 + 11X^9 + 25X^{10} + \cdots,$$

which yields

(a) $P(E) < 1.2118 \times 10^{-4}$ for $p = 0.01$,
(b) $P(E) < 7.7391 \times 10^{-8}$ for $p = 0.001$.

The bit error probability is bounded by (12.29)

$$P_b(E) < \sum_{d=d_{free}}^{\infty} B_d P_d < B(X)\big|_{X=2\sqrt{p(1-p)}} = \frac{1}{k}\left.\frac{\partial A(W,X)}{\partial W}\right|_{X=2\sqrt{p(1-p)},W=1}.$$

From Example 11.12,

$$A(W,X) = \frac{WX^7 + W^2(X^6 - X^8)}{1-W(2X+X^3)} = WX^7 + W^2\left(X^6 + X^8 + X^{10}\right) + W^3\left(2X^7 + 3X^9 + 3X^{11} + X^{13}\right) + \cdots.$$

Hence,

$$\frac{\partial A(W,X)}{\partial W} = \frac{X^7 + 2W(X^6 - 3X^8 - X^{10}) - 3W^2(2X^7 - X^9 - X^{11})}{(1-2WX-WX^3)^2} + \cdots$$

and

$$\left.\frac{\partial A(W,X)}{\partial W}\right|_{W=1} = \frac{2X^6 - X^7 - 2X^8 + X^9 + X^{11}}{(1-2X-X^3)^2} = 2X^6 + 7X^7 + 18X^8 + \cdots.$$

This yields

(a) $P_b(E) < 3.0435 \times 10^{-4}$ for $p = 0.01$,

(b) $P_b(E) < 1.6139 \times 10^{-7}$ for $p = 0.001$.

12.11 The event error probability is given by (12.26)

$$P(E) \approx A_{d_{free}} \left[2\sqrt{p(1-p)}\right]^{d_{free}} \approx A_{d_{free}} 2^{d_{free}} p^{d_{free}/2}$$

and the bit error probability (12.30) is given by

$$P_b(E) \approx B_{d_{free}} \left[2\sqrt{p(1-p)}\right]^{d_{free}} \approx B_{d_{free}} 2^{d_{free}} p^{d_{free}/2}.$$

From Problem 12.10,

$$d_{free} = 6, \quad A_{d_{free}} = 1, \quad B_{d_{free}} = 2.$$

(a) For $p = 0.01$,

$$P(E) \approx 1 \cdot 2^6 \cdot (0.01)^{6/2} = 6.4 \times 10^{-5}$$
$$P_b(E) \approx 2 \cdot 2^6 \cdot (0.01)^{6/2} = 1.28 \times 10^{-4}.$$

(b) For $p = 0.001$,

$$P(E) \approx 1 \cdot 2^6 \cdot (0.001)^{6/2} = 6.4 \times 10^{-8}$$
$$P_b(E) \approx 2 \cdot 2^6 \cdot (0.001)^{6/2} = 1.28 \times 10^{-7}.$$

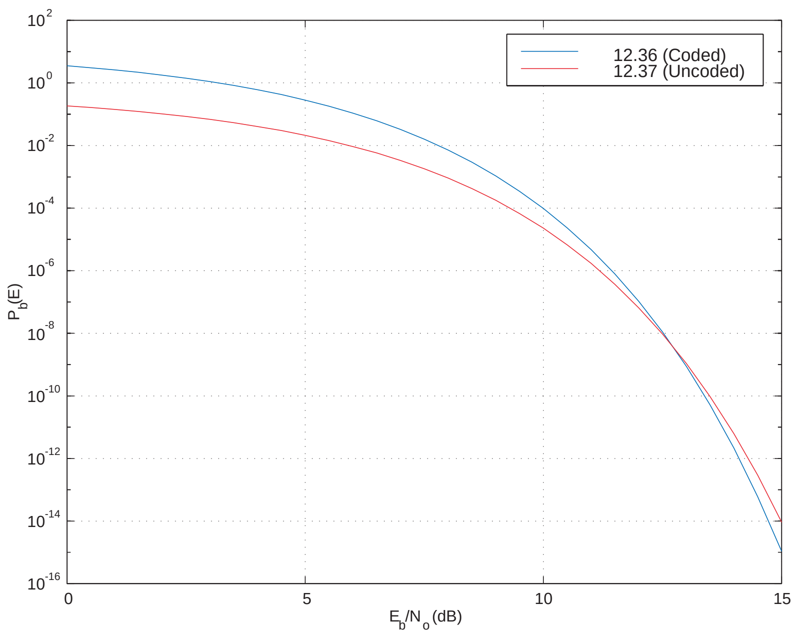
12.12 The $(3,1,2)$ encoder of (12.1) has $d_{free} = 7$ and $B_{d_{free}} = 1$. Thus, expression (12.36) becomes

$$P_b(E) \approx B_{d_{free}} 2^{d_{free}/2} e^{-(R d_{free}/2)\cdot(E_b/N_o)} = 2^{(7/2)} e^{-(7/6)\cdot(E_b/N_o)}$$

and (12.37) remains

$$P_b(E) \approx \frac{1}{2} e^{-E_b/N_o}.$$

These expressions are plotted versus E_b/N_o in the figure below.

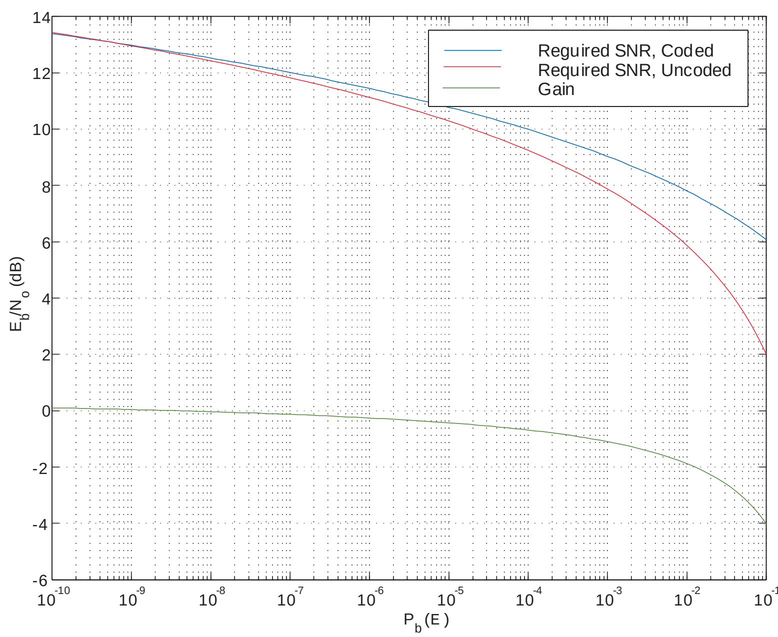


12.36 (Coded)
12.37 (Uncoded)
$P_b(E)$
E_b/N_o (dB)
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
0
5
10
15

Equating the above expressions and solving for E_b/N_o yields

$$\begin{aligned} 2^{(7/2)}e^{-(7/6)\cdot(E_b/N_o)} &= \frac{1}{2}e^{-E_b/N_o} \\ 1 &= 2^{(9/2)}e^{-(1/6)(E_b/N_o)} \\ e^{(-1/6)(E_b/N_o)} &= 2^{-(9/2)} \\ (-1/6)(E_b/N_o) &= \ln(2^{-(9/2)}) \\ E_b/N_o &= -6\ln(2^{-(9/2)}) = 18.71, \end{aligned}$$

which is $E_b/N_o = 12.72$dB, the coding threshold. The coding gain as a function of $P_b(E)$ is plotted below.



Note that in this example, a short constraint length code ($\nu = 2$) with hard decision decoding, the approximate expressions for $P_b(E)$ indicate that a positive coding gain is only achieved at very small values of $P_b(E)$, and the asymptatic coding gain is only 0.7dB.

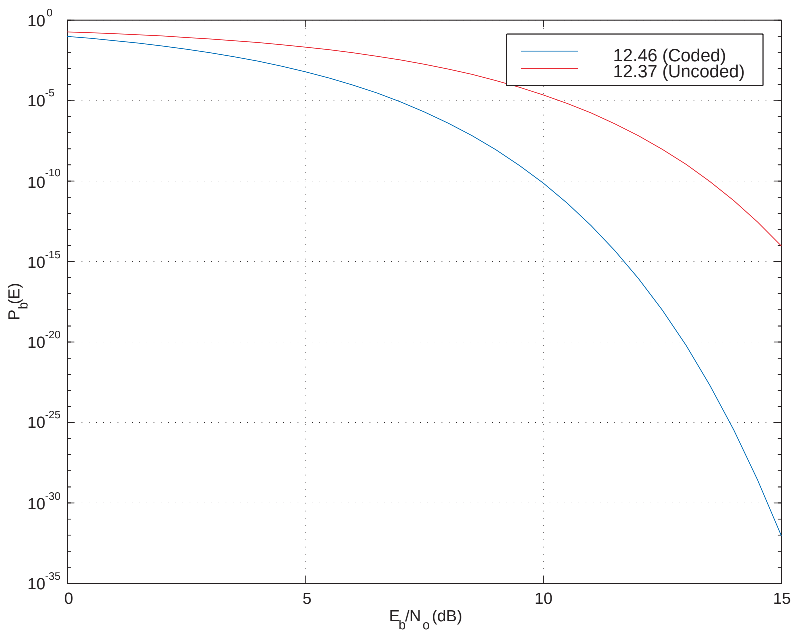
12.13 The $(3, 1, 2)$ encoder of Problem 12.1 has $d_{free} = 7$ and $B_{d_{free}} = 1$. Thus, expression (12.46) for the unquantized AWGN channel becomes

$$P_b(E) \approx B_{d_{free}} e^{-Rd_{free}E_b/N_o} = e^{-(7/3)\cdot(E_b/N_o)}$$

and (12.37) remains

$$P_b(E) \approx \frac{1}{2}e^{-E_b/N_o}.$$

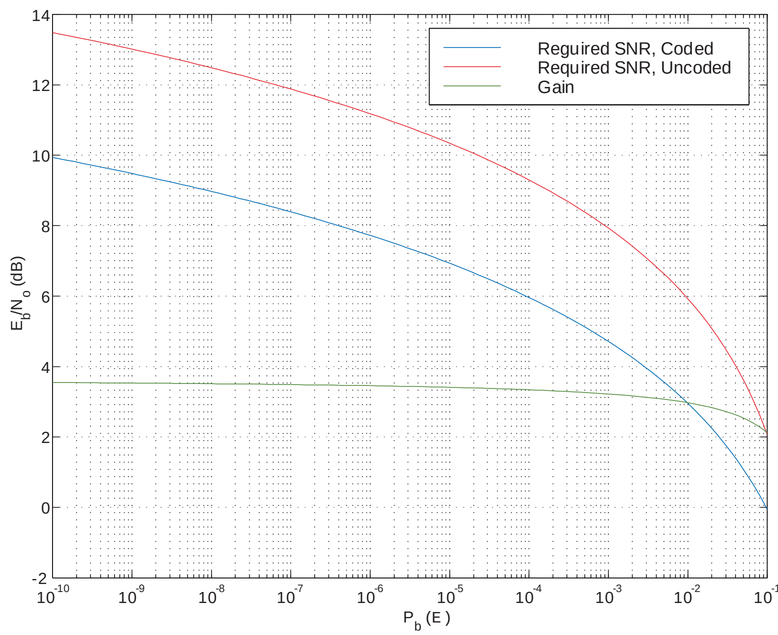
These expressions are plotted versus E_b/N_o in the figure below.



Equating the above expressions and solving for E_b/N_o yields

$$\begin{aligned} e^{-(7/3)\cdot(E_b/N_o)} &= \frac{1}{2}e^{-E_b/N_o} \\ 1 &= \frac{1}{2}e^{(4/3)(E_b/N_o)} \\ e^{(4/3)(E_b/N_o)} &= 2 \\ (4/3)(E_b/N_o) &= \ln(2) \\ E_b/N_o &= (3/4)\ln(2) = 0.5199, \end{aligned}$$

which is $E_b/N_o = -2.84dB$, the coding threshold. (Note: If the slightly tighter bound on $Q(x)$ from (1.5) is used to form the approximate expressin for $P_b(E)$, the coding threshold actually moves to $-\infty\, dB$. But this is just an artifact of the bounds, which are not tight for small values of E_b/N_o.) The coding gain as a function of $P_b(E)$ is plotted below. Note that in this example, a short constraint length code ($\nu = 2$) with soft decision decoding, the approximate expressions for $P_b(E)$ indicate that a coding gain above 3.0 dB is achieved at moderate values of $P_b(E)$, and the asymptotic coding gain is 3.7 dB.



Required SNR, Coded
Required SNR, Uncoded
Gain
E_b/N_o (dB)
P_b (E)
14
12
10
8
6
4
2
0
-2
10^{-10}
10^{-9}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

12.14 The IOWEF function of the $(3,1,2)$ encoder of (12.1) is

$$A(W,X) = \frac{WX^7}{1 - WX - WX^3}$$

and thus (12.39b) becomes

$$P_b(E) < B(X)|_{X=D_o} = \frac{1}{k}\left.\frac{\partial A(W,X)}{\partial W}\right|_{X=D_0,W=1} = \left.\frac{X^7}{(1-WX-WX^3)^2}\right|_{X=D_0,W=1}.$$

For the DMC of Problem 12.4, $D_0 = 0.42275$ and the above expression becomes

$$P_b(E) < 9.5874 \times 10^{-3}.$$

If the DMC is converted to a BSC, then the resulting crossover probability is $p = 0.091$. Using (12.29) yields

$$P_b(E) < B(X)|_{X=D_o} = \frac{1}{k}\left.\frac{\partial A(W,X)}{\partial W}\right|_{X=2\sqrt{p(1-p)},W=1} = \left.\frac{X^7}{(1-WX-WX^3)^2}\right|_{X=2\sqrt{p(1-p)},W=1} = 3.7096\times 10^{-1},$$

about a factor of 40 larger than the soft decision case.

12.16 For the optimum $(2,1,7)$ encoder in Table 12.1(c), $d_{free} = 10$, $A_{d_{free}} = 1$, and $B_{d_{free}} = 2$.

(a) From Table 12.1(c)

$$\gamma = 6.99dB.$$

(b) Using (12.26) yields

$$P(E) \approx A_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 1.02 \times 10^{-7}.$$

(c) Using (12.30) yields

$$P_b(E) \approx B_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 2.04 \times 10^{-7}.$$

(d) For this encoder

$$\mathbf{G}^{-1} = \left[\begin{array}{c} D^2 \\ 1 + D^+ D^2 \end{array}\right]$$

and the amplification factor is $A = 4$.

For the quick-look-in $(2,1,7)$ encoder in Table 12.2, $d_{free} = 9$, $A_{d_{free}} = 1$, and $B_{d_{free}} = 1$.

(a) From Table 12.2

$$\gamma = 6.53dB.$$

(b) Using (12.26) yields

$$P(E) \approx A_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 5.12 \times 10^{-7}.$$

(c) Using (12.30) yields

$$P_b(E) \approx B_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 5.12 \times 10^{-7}.$$

(d) For this encoder

$$\mathbf{G}^{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the amplification factor is $A = 2$.

12.17 The generator matrix of a rate $R = 1/2$ systematic feedforward encoder is of the form

$$\mathbf{G} = \begin{bmatrix} 1 & \mathbf{g}^{(1)}(D) \end{bmatrix}.$$

Letting $\mathbf{g}^{(1)}(D) = 1 + D + D^2 + D^5 + D^7$ achieves $d_{free} = 6$ with $B_{d_{free}} = 1$ and $A_{d_{free}} = 1$.

(a) The soft-decision asymptotic coding gain is

$$\gamma = 4.77dB.$$

(b) Using (12.26) yields

$$P(E) \approx A_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 6.4 \times 10^{-5}.$$

(c) Using (12.30) yields

$$P_b(E) \approx B_{d_{free}} 2^{d_{free}} p^{d_{free}/2} = 6.4 \times 10^{-5}.$$

(d) For this encoder (and all systematic encoders)

$$\mathbf{G}^{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the amplification factor is $A = 1$.

12.18 The generator polynomial for the $(15, 7)$ BCH code is

$$\mathbf{g}(X) = 1 + X^4 + X^6 + X^7 + X^8$$

and $d_g = 5$. The generator polynomial of the dual code is

$$\mathbf{h}(X) = \frac{X^{15} + 1}{X^8 + X^7 + X^6 + X^4 + 1} = X^7 + X^6 + X^4 + 1$$

and hence $d_h \geq 4$.

(a) The rate $R = 1/2$ code with composite generator polynomial $\mathbf{g}(D) = 1 + D^4 + D^6 + D^7 + D^8$ has generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 + D^2 + D^3 + D^4 & D^3 \end{bmatrix}$$

and $d_{free} \geq \min(5, 8) = 5$.

(b) The rate $R = 1/4$ code with composite generator polynomial $\mathbf{g}(D) = \mathbf{g}(D^2) + D\mathbf{h}(D^2) = 1 + D + D^8 + D^9 + D^{12} + D^{13} + D^{14} + D^{15} + D^{16}$ has generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 + D^2 + D^3 + D^4 & 1 + D^2 + D^3 & D^3 & D^3 \end{bmatrix}$$

and $d_{free} \geq \min(d_g + d_h, 3d_g, 3d_h) = \min(9, 15, 12) = 9$.

The generator polynomial for the $(31, 16)$ BCH code is

$$\mathbf{g}(X) = 1 + X + X^2 + X^3 + X^5 + X^7 + X^8 + X^9 + X^{10} + X^{11} + X^{15}$$

and $d_g = 7$. The generator polynomial of the dual code is

$$\mathbf{h}(X) = \frac{X^{15} + 1}{\mathbf{g}(X)} = X^{16} + X^{12} + X^{11} + X^{10} + X^9 + X^4 + X + 1$$

and hence $d_h \geq 6$.

(a) The rate $R = 1/2$ code with composite generator polynomial $\mathbf{g}(D) = 1 + D + D^2 + D^3 + D^5 + D^7 + D^8 + D^9 + D^{10} + D^{11} + D^{15}$ has generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 + D + D^4 + D^5 & 1 + D + D^2 + D^3 + D^4 + D^5 + D^7 \end{bmatrix}$$

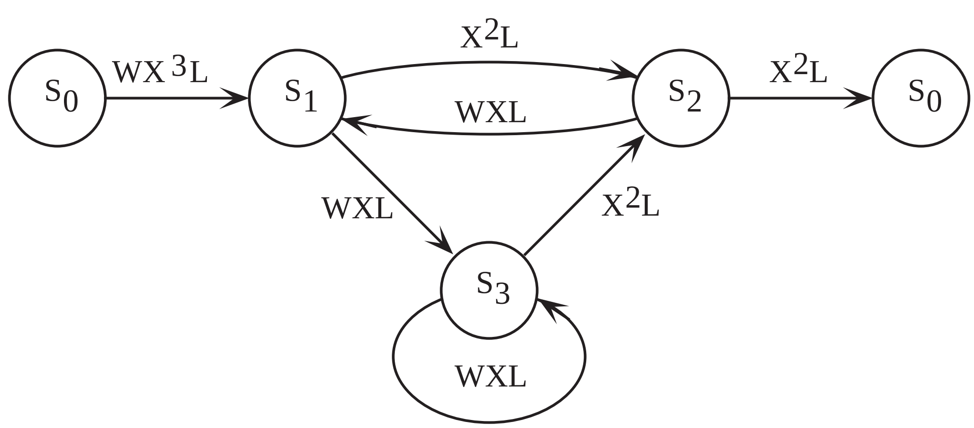
and $d_{free} \geq \min(7, 12) = 7$.

(b) The rate $R = 1/4$ code with composite generator polynomial $\mathbf{g}(D) = \mathbf{g}(D^2) + D\mathbf{h}(D^2) = 1 + D + D^2 + D^3 + D^4 + D^6 + D^9 + D^{10} + D^{14} + D^{16} + D^{18} + D^{19} + D^{20} + D^{21} + D^{22} + D^{23}$ has generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 + D + D^4 + D^5 & 1 + D^2 + D^5 + D^6 + D^8 & 1 + D + D^2 + D^3 + D^4 + D^5 + D^7 & 1 + D^5 \end{bmatrix}$$

and $d_{free} \geq \min(d_g + d_h, 3d_g, 3d_h) = \min(13, 21, 18) = 13$.

12.20 (a) The augmented state diagram is shown below.



The generating function is given by

$$A(W, X, L) = \frac{\sum_i F_i \Delta_i}{\Delta}.$$

There are 3 cycles in the graph:

$$\begin{array}{lll} \text{Cycle 1:} & S_1S_2S_1 & C_1 = WX^3L^2 \\ \text{Cycle 2:} & S_1S_3S_2S_1 & C_2 = W^2X^4L^3 \\ \text{Cycle 3:} & S_3S_3 & C_3 = WXL. \end{array}$$

There is one pair of nontouching cycles:

$$\text{Cycle pair 1:} \quad \text{(loop 1, loop 3)} \quad C_1C_3 = W^2X^4L^3.$$

There are no more sets of nontouching cycles. Therefore,

$$\begin{aligned} \Delta &= 1 - \sum_i C_i + \sum_{i',j'} C_{i'}C_{j'} \\ &= 1 - (WX^3L^2 + W^2X^4L^3 + WXL) + W^2X^4L^3. \end{aligned}$$

There are 2 forward paths:

$$\begin{array}{lll} \text{Forward path 1:} & S_0S_1S_2S_0 & F_1 = WX^7L^3 \\ \text{Forward path 2:} & S_0S_1S_3S_2S_0 & F_2 = W^2X^8L^4. \end{array}$$

Only cycle 3 does not touch forward path 1, and hence

$$\Delta_1 = 1 - WXL.$$

Forward path 2 touches all the cycles, and hence

$$\Delta_2 = 1.$$

Finally, the WEF $A(W, X, L)$ is given by

$$A(W, X, L) = \frac{WX^7L^3(1 - WXL) + W^2X^8L^4}{1 - (WX^3L^2 + W^2X^4L^3 + WXL) + W^2X^4L^3} = \frac{WX^7L^3}{1 - WXL - WX^3L^2}$$

and the generating WEF's $A_i(W, X, L)$ are given by:

$$\begin{aligned} A_1(W, X, L) &= \frac{WX^3L(1 - WXL)}{\Delta} = \frac{WX^3L(1 - WXL)}{1 - WXL - WX^3L^2} \\ &= W\,X^3\,L + W^2\,X^6\,L^3 + W^3\,X^7\,L^4 + (W^3X^9 \; + W^4X^8\,)\,L^5 + (2\,W^4, X^{10} \; + W^5X^9\,)\,L^6 + \cdots \\ A_2(W, X, L) &= \frac{WX^5L^2(1 - WXL) + W^2X^6L^3}{\Delta} = \frac{WX^5L^2}{1 - WXL - WX^3L^2} \\ &= W\,X^5\,L^2 + W^2\,X^6\,L^3 + (W^2\,X^8 + W^3\,X^7)\,L^4 + (2\,W^3\,X^9 + W^4\,X^8)\,L^5 \\ &\quad +(W^3\,X^{11} + 3\,W^4\,X^{10} + W^5\,X^9)L^6 + \cdots \\ A_3(W, X, L) &= \frac{W^2X^4L^2}{\Delta} = \frac{W^2X^4L^2}{1 - WXL - WX^3L^2} \\ &= W^2\,X^4\,L^2 + W^3\,X^5\,L^3 + (W^3\,X^7 + W^4\,X^6)\,L^4 + (2\,W^4\,X^8 + W^5\,X^7)\,L^5 \\ &\quad +(W^4\,X^{10} + 3\,W^5\,X^9 + W^6\,X^8)L^6 \cdots \end{aligned}$$

(b) This code has $d_{free} = 7$, so $\boldsymbol{\tau}_{min}$ is the minimum value of $\boldsymbol{\tau}$ for which $d(\boldsymbol{\tau}) = d_{free} + 1 = 8$. Examining the series expansions of $A_1(W, X, L)$, $A_2(W, X, L)$, and $A_3(W, X, L)$ above yields $\boldsymbol{\tau}_{min} = 5$.

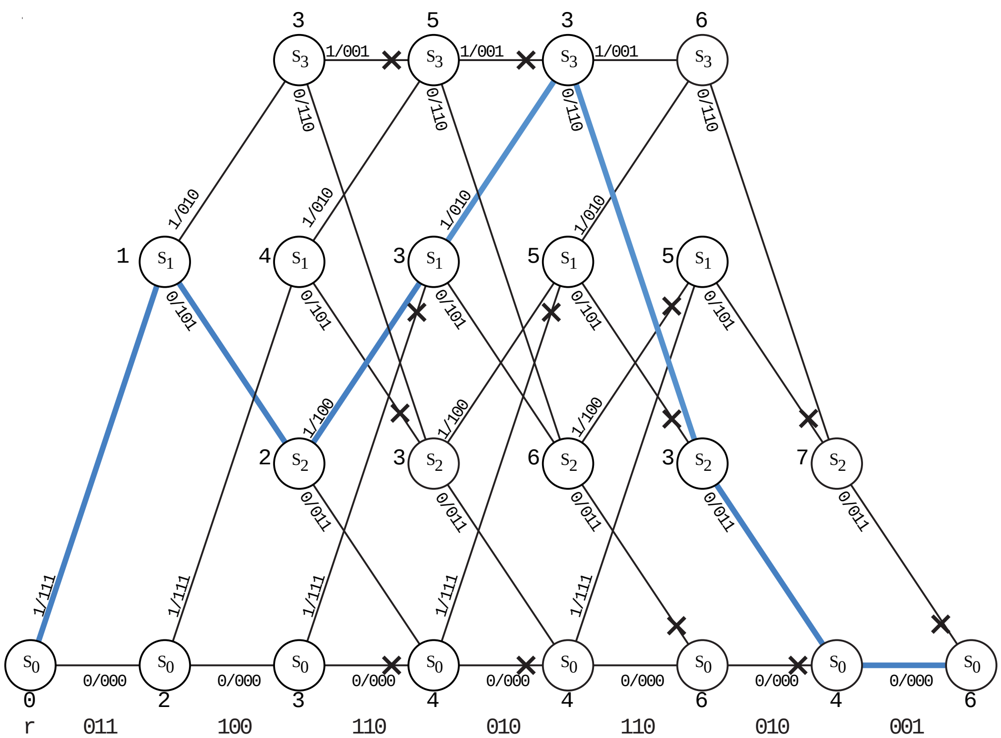
(c) A table of $d(\boldsymbol{\tau})$ and $A_{d(\boldsymbol{\tau})}$ is given below.

$\boldsymbol{\tau}$	$d(\boldsymbol{\tau})$	$A_{d(\boldsymbol{\tau})}$
0	3	1
1	4	1
2	5	1
3	6	1
4	7	1
5	8	1

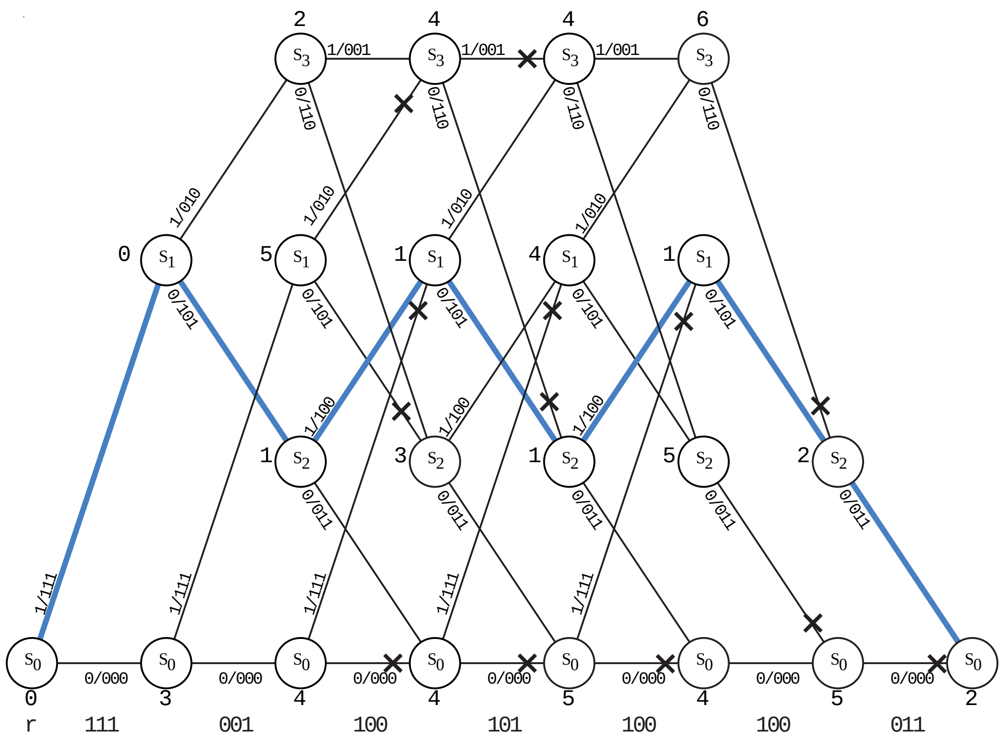
(d) From part (c) and by looking at the series expansion of $A_3(W, X, L)$, it can be seen that

$$\lim_{\boldsymbol{\tau}\to\infty} d(\boldsymbol{\tau}) = \boldsymbol{\tau} + 3.$$

12.21 For a BSC, the trellis diagram of Figure 12.6 in the book may be used to decode the three possible 21-bit subequences using the Hamming metric. The results are shown in the three figures below. Since the **r** used in the middle figure (b) below has the smallest Hamming distance (2) of the three subsequences, it is the most likely to be correctly synchronized.

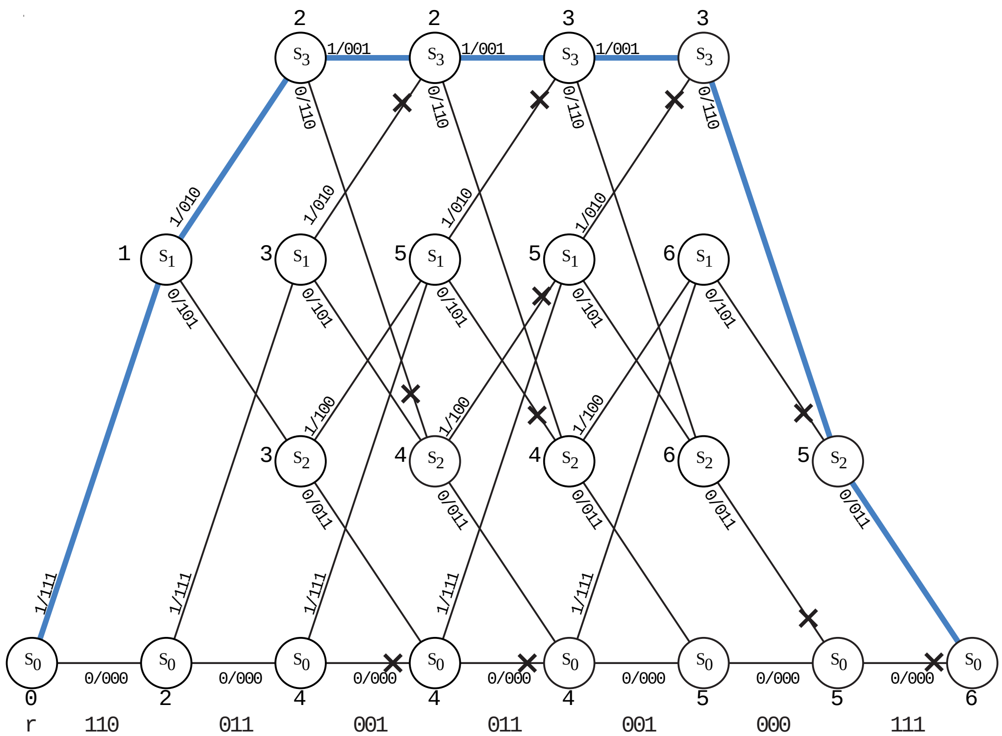


(a)



2
4
4
6
1/001
0/110
1/010
0/101
1/100
0/011
1/111
0/000
0
5
1
4
1
1
3
1
5
2
0
3
4
4
5
4
5
2
r
111
001
100
101
100
100
011

(b)


2 2 3 3
s_3 1/001 s_3 1/001 s_3 1/001 s_3
0/110
1/010
1 s_1 3 s_1 5 s_1 5 s_1 6 s_1
0/101
1/100
3 s_2 4 s_2 4 s_2 6 s_2 5 s_2
0/011
1/111
s_0 0/000 s_0 0/000 s_0 0/000 s_0 0/000 s_0 0/000 s_0 0/000 s_0 0/000 s_0
0 2 4 4 4 5 5 6
r 110 011 001 011 001 000 111

(c)