Modeling a Servo Motor System

Definitions

- ***Motor:*** A device that receives a continuous (Analog) signal and operates continuously in time.
- ***Digital Controller:*** Discretizes the amplitude of the signal and also operates at discrete time (sample data).

Continued…

- ***Position Sensor:*** Operates continuously in time but discretizes the amplitude.
- ***Power Amplifier:*** Power amplifier, which produces a continuous signal but operates at discrete times.

Elements to be modeled

- Amplifier
- Motor & Load
- Position Sensor
- Controller

Amplifier Motor Modeling

Current or Torque Mode Amplifier

- In this type of operation mode, amplifier output current, *I* that is directly proportional to the input voltage *V* the proportionality factor K_a

$$I = K_a V \longrightarrow (1)$$

- Torque $T_g = K_t I \longrightarrow (2)$ where $K_t \rightarrow torque\ constant$

Moment of Inertia of Solid Disc

- mass ‘m’ and radius ‘r’:

$$J = \frac{mr^2}{2}$$

$$Torque = Force \; x \; Liver \; Arm$$

$$T = Fr$$

- If J is total moment of inertia of load & motor, T_f is opposing friction,

$$J\alpha = T_f + T_g \longrightarrow (3)$$

Continued....

- Where α is acceleration,

$$\alpha = \frac{d\omega}{dt}$$

- Where ω *is the velocity,*

$$\omega = \int \alpha \, dt \rightarrow Taking\ Laplace\ Transform \rightarrow \omega = \frac{1}{s}\,\alpha \quad (4)$$

$$\omega = \frac{d\theta}{dt} \rightarrow \theta = \int \omega \, dt \rightarrow Taking\ Laplace\ Transform \rightarrow \theta = \frac{1}{s}\,\omega \quad (5)$$

Combining equations 1 through 5

$$\theta = \frac{1}{s}\omega \rightarrow = \left(\frac{1}{s}\right)\left(\frac{1}{s}\alpha\right) \rightarrow = \frac{1}{s^2}\frac{T_g}{J}$$

$$if\ T_f\ is\ neglected \rightarrow = \frac{1}{s^2}\frac{K_t I}{J}$$

$$\theta = \frac{1}{s^2}\frac{K_a K_t V}{J}$$

$$\frac{\theta}{V} = \frac{K_a K_t}{Js^2} \text{-----} (6)$$

Position Sensor Modeling

Position Sensor Modeling

- Motor position is indicated by position sensor as signal ‘c’.
- K_f proportionality factor, K_f equals the number of units of feedback per one radian of rotation.
- Encoder provides the position, suppose an incremental encoder generates N pulses per revolution, that the encoder generates output.

Continued…

- Channels A & B produces 1000 pulses for each encoder rotation.
- As two signals are shifted by one quarter of a cycle, the controller can divide each encoder cycle into four quadrant counts resulting in an effective resolution of 4N counts per revolution or turn. Since each revolution is 2π radians, the resulting encoder gain is

$$K_f = \frac{4N}{2\pi} \longrightarrow (7)$$

Model for Incremental Encoder:

$$K_f = \frac{4N}{2\pi} \longrightarrow \textcircled{7}$$

$$= \frac{4\ x\ 1000}{2\pi} = 636.537\,\frac{counts}{rev}$$

- Thus, 1000 pulse per rev encoder has an equivalent gain of 636 counts/ revolution.

- Another common type of position sensor is the one of binary representation. Total number of positions per revolution is the model for this sensor is:

$$K_f = \frac{2^n}{2\pi} \quad \text{-----} \quad (8)$$

- For example- For Absolute encoder or resolver with 16-Bit binary position signal has a gain of: $K_f = \frac{2^{16}}{2\pi} = 10{,}435\ counts/radians$

SCHOOL of TECHNOLOGY

MECHATRONICS ENGINEERING TECHNOLOGY

PURDUE UNIVERSITY CALUMET

Modeling a Controller

Modeling a Controller

- The desired position signal is R(t) or simply 'R' actual position is 'C'. Thus position error 'E':

$$E = R - C$$

- This position error is used to generate the output signal that drives the motor.

Continued…

- The proportional term x_p ,

$$x_p = (P)(E)$$

Error, input

Gain of the proportional part of the controller

Continued…

- The derivative term x_d ,

$$x_d = (sD)(E)$$

Gain of derivative controller → sD

Error, input → E

- The integral term x_i ,

$$x_i = \left(\frac{1}{s} I\right) E$$

Gain of the Integral controller — Error, input

- Sum of all three outputs,

$$x = x_p + x_d + x_i$$

- The Transfer function F(s), relating the output ‘x’ to position error E is,

$$F(s) = \frac{x}{E} = P + sD + \frac{1}{s} I - - - - - - PID\ Controller$$

Examples

- Example- Digital to Analog converter resolution, 8-16 bit. DAC having output voltage range -10 V to +10 V:

Solution

Output- -10 V to +10 V

Gain of DAC 'K', equal to the number of volts it produces per unit of 'x' input signal.

- DAC Resolution in ‘n’ bits, the DAC Gain equals:

A 12-Bit DAC has K=0.0048 V/unit

$$K = \frac{20}{2^n} = \frac{20}{2^{12}} = 0.0048\ V/unit$$

SCHOOL of TECHNOLOGY

$$\omega = \frac{1}{s}\alpha \;\Rightarrow\; \theta = \frac{1}{s}\omega \;\Rightarrow\; \theta = \frac{1}{s}\frac{1}{s}\alpha \;\Rightarrow\; \theta = \frac{1}{s^2}\frac{T_g}{J}$$

$$= \frac{1}{s^2}\frac{K_t I}{J}$$

$$= \frac{1}{s^2}K_t K_a \frac{V}{J}$$

$$\frac{\theta}{V} = \frac{1}{s^2}\frac{K_t K_a}{J}$$

$$\frac{\omega}{U} = \frac{K_t}{s^2 JL + sJr + K_e K_t}$$

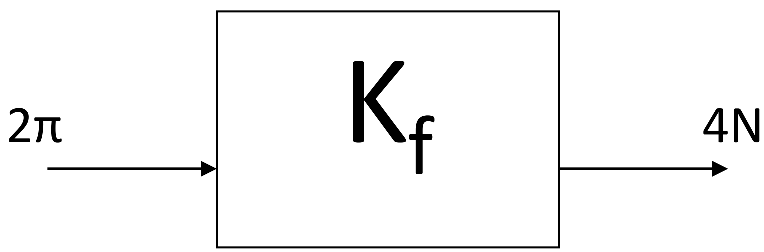
NSF

MECHATRONICS ENGINEERING TECHNOLOGY

PURDUE UNIVERSITY CALUMET

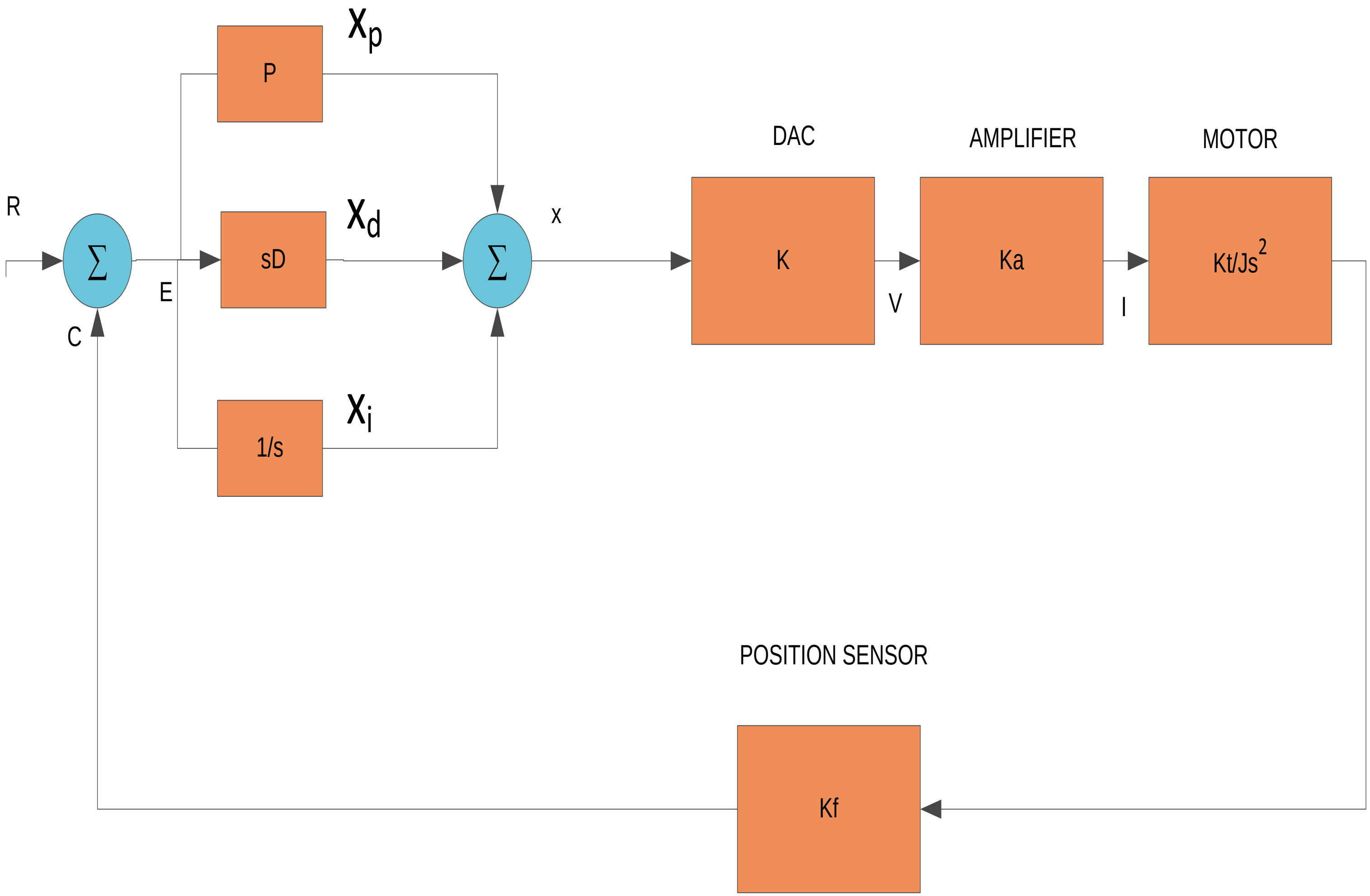
Encoder Gain

$$K_f = \frac{4N}{2\pi}$$



$$\left(\frac{4\ x\ 1000}{(2)(3.1428)}\right) = 636\ counts/rad$$

A Servo Motor System



System with Voltage Amplifier

- There are amplifier that are designed to produce a proportional output, N, rather than current, I. In this case, the amplifier is modeled as a voltage gain K_v

$$U = K_v V \longrightarrow (16)$$

- When the voltage U is applied to the motor, it produces a current, I, which depends on the motor velocity angular velocity ω. The circuit equation of the motor is

$$U = rI + sLI + K_e \omega \longrightarrow (17)$$

Note

Motor Voltage includes three terms that represents three physical effects:

- rI, represents the voltage across the resistance, r.
- sLI, represents the voltage across the inductance, L.
- K_e ω, represents the emf indices by the motor that is function of 'ω', angular velocity.

- The dynamic equation 3 can be represented in terms of ω as follows:

$$J_{\alpha} + T_f = T_g$$

- Since from equation 4 $\omega = (1/s\ \alpha)$ therefore $\alpha = s\omega$

$$Js\omega + T_f = T_g \rightarrow (18)$$

- Thus we can that dynamic behavior of the motor depends in the operation mode of the amplifier as evidence from the models.

- Combining equation 2 and equation 18 and neglecting T_f, friction factor

$$Js\omega = T_g$$

$$Js\omega = K_t\, I$$

$$I = (1/K_t)\,(Js\omega) \text{---} (19)$$

Combining equation 17 and equation 19

$$U = 1/K_t\,(s^2 J\omega L + sJr\omega + K_e K_t \omega) \text{---} (20)$$

MECHATRONICS ENGINEERING TECHNOLOGY

PURDUE UNIVERSITY CALUMET

SCHOOL of TECHNOLOGY

Now factoring ω

$$M(s) = \omega/U = K_t\ (s^2JL + SJr + K_eK_t) \quad (21)$$

Or,

$$M(s) = 1/(K_e(sT_m + 1)\ (sT_e + 1)) \quad (22)$$

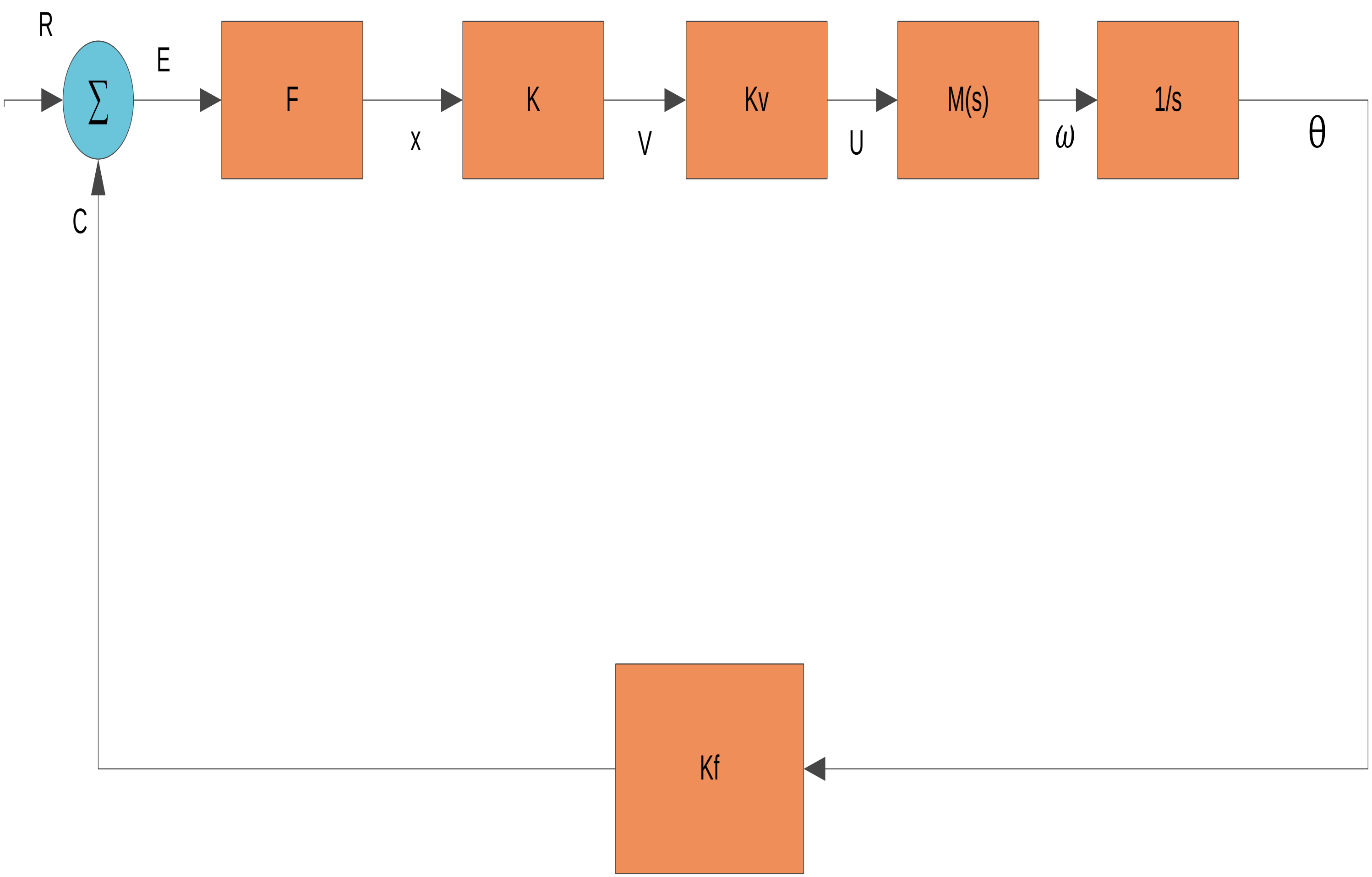
Where, $T_m = J_r\ /\ (K_e\ K_t)$ and $T_e = L/r$ (23)

SCHOOL of TECHNOLOGY

MECHATRONICS ENGINEERING TECHNOLOGY

PURDUE UNIVERSITY CALUMET

BLOCK DIAGRAM



- The overall transfer function representing the combined effect of the motor and the amplifier is derived by combining the equation 5, 16 and 22 which is as follows:

$$\Theta = (1/s)\ \omega \text{---} (5)$$

$$U = K_v V \text{---} (16)$$

$$M(s) = 1/(K_e(sT_m + 1)(sT_e + 1)) \text{---} (22)$$

$$\Theta/V = K_v / K_e s((sT_m + 1)(sT_e + 1)) \text{-----} (24)$$

EXAMPLE 1

- ***Amplifier:*** Operating in the current node with the current gain K_a of 0.6 amp per volt.
- ***Motor- load:*** Total amount of inertia, $J = 2x10^{-4}$ $kg.m^2$ and torque constant $K_t = 0.12$ Nm/A.
- ***Position Sensor:*** The position sensor is an incremental encoder with 1000 lines per revolution producing a resolution of 4000 counts/revolution.
- ***Motion Controller:*** The motion controller has a 14-bit DAC, and the filter parameter are P=20 and D=0.2.

MECHATRONICS
ENGINEERING
TECHNOLOGY

PURDUE
UNIVERSITY
CALUMET

MATHEMATICAL MODEL

- $\Theta/V = K_a K_t / Js^2 = (0.6\ A/V)\ (0.12\ Nm/A) / (2x10^{-4}\ kg.m^2)\ s^2 = 360/s^2$
- The incremental position sensor, according to equation 7 is modeled as:

$$K_f = 4N/2\pi \approx 636 \text{ counts/radian}$$

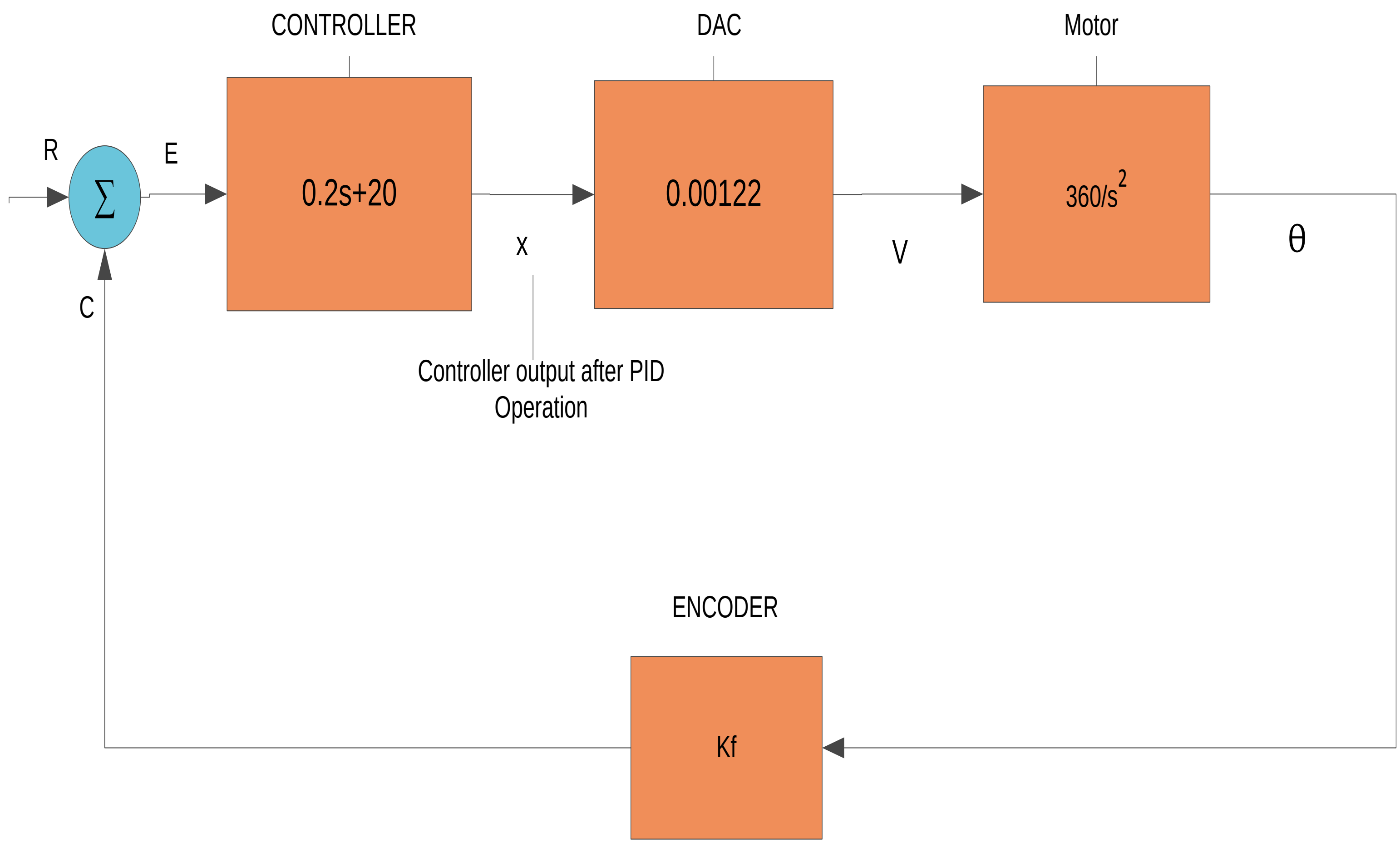
- The gain of the DAC, K, is given by the equation 14 as:

$$K = 20/2^{14} = 0.00122 \text{ Volts/Count}$$

- Motion controller sensor is given by the equation 10 and 12

$$X/E = P + sD = 20+0.2s$$

Complete Mathematical Model



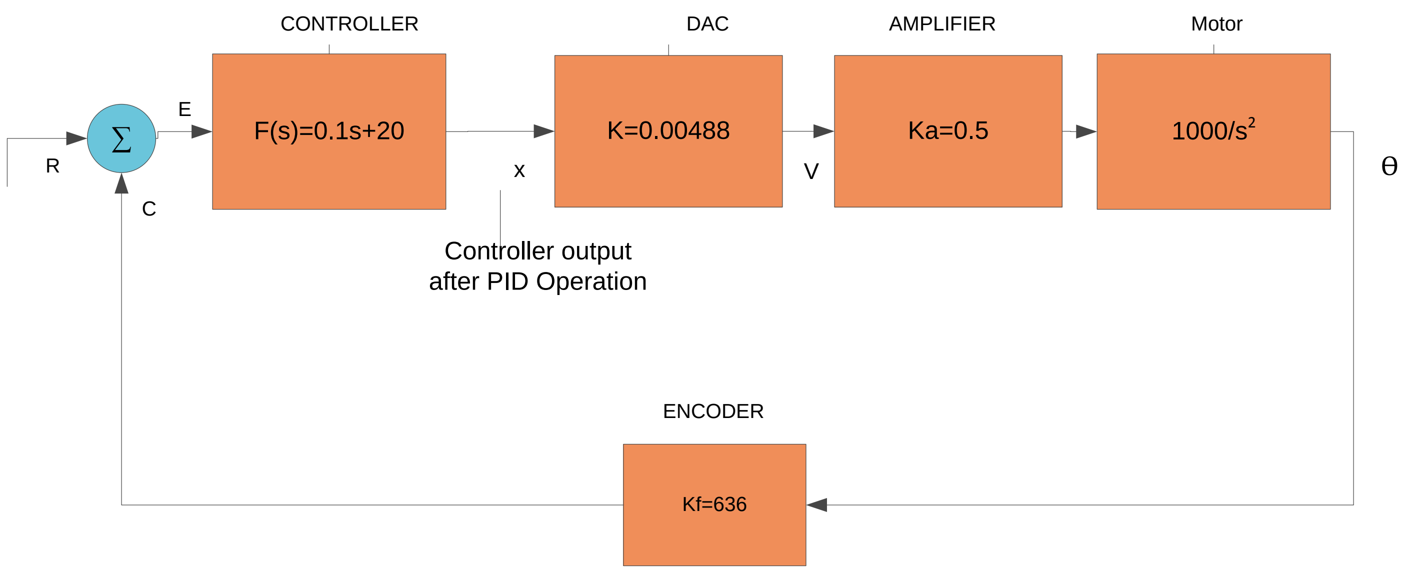
EXAMPLE 2

Parameter	Definition
K_t = 0.1 Nm/A	Motor Torque Constant
J = 10^{-4} Kg.m^2	Moment of Inertia
K_a = 0.5 A/V	Amplifier Gain
K_f = 636 Counts/radian	Encoder Gain
K = 0.00488 volts/count	DAC Gain
P = 20 (proportional constant)	
D = 0.1 (Derivative constant)	

Solution

- Transfer Function of the Controller

$$F(s) = 20+0.1s$$



Open-loop Transfer Function L(s)

- $L(s) = (\Theta/V)\ (K_f)\ (X/E)\ (K)$

  $= (K_a K_t / Js^2)\ (K_f)\ (P + sD)\ (K)$

  $= (1000/s^2)\ (636)\ (0.00488)\ (0.1s + 20)$

- $L(s) = (3103.68/s^s)\ (0.1s+20)$

$$\mathbf{L(s) = (310.368s + 62073.6)/s^2}$$

MECHATRONICS ENGINEERING TECHNOLOGY

PURDUE UNIVERSITY CALUMET