Optimal decentralized control for vehicle formation

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Abstract

In this paper, an incrementally linear decentralized control law is proposed for the formation of cooperative vehicles with leader-follower topology. It is assumed that each vehicle knows the modeling parameters of other vehicles with uncertainty as well as the expected values of their initial states. A decentralized control law is proposed, which aims to perform as close as possible to a centralized LQR controller. It is shown that the decentralized controller behaves as the same as its centralized counterpart, provided a priori information of each vehicle about others is accurate. Since this condition does not hold in practice, a method is presented to evaluate the deviation of the performance of the decentralized system from that of its centralized counterpart. Furthermore, the necessary and sufficient conditions for the stability of the overall closed-loop system in presence of parameter perturbations are given through a series of simple tests. It is shown that stability of the overall system is independent of the magnitude of the expected value of the initial states. Moreover, it is shown that the decentralized control system is likely to be more robust than the centralized one. Optimal decentralized cheap control problem is then investigated for the leader-follower formation structure, and a closed form solution is given for the case when the system parameters meet a certain condition. Simulation results demonstrate the effectiveness of the proposed controller in terms of feasibility and performance.

1. INTRODUCTION

In the past several years, a certain class of interconnected systems, namely acyclic systems, has found applications in different practical problems such as formation flight, underwater vehicles, automated highway, robotics, satellite constellation, etc., which have leader-follower structures or structures with virtual leaders [1-13]. The main feature of this class of systems is that their structural graphs are acyclic, i.e. they do not have any directed cycles.

In a leader-follower formation structure, each vehicle is provided with some information (e.g., acceleration or velocity) of certain set of vehicles. It is shown in the literature that the control problem of such formation can be formulated as the decentralized control problem of an acyclic interconnected system, where each local controller uses only the information of its corresponding subsystem (e.g., see [2]). The objective of this paper is to design a decentralized controller which stabilizes any system with an acyclic structure, and performs sufficiently close to the optimal centralized controller. In other words, it is desired to reduce the degradation of the performance due to the information flow constraints in decentralized control structures.

During the past three decades, much effort has been made to formulate the optimal decentralized control problem, or solve it numerically. The main objective is to find a decentralized feedback law for an interconnected system in order to attain a sufficiently small performance index. These works can mainly be categorized as follows:

1. The first approach is to eliminate all of the interconnections between the subsystems in order to obtain a set of decoupled subsystems, and then design a local optimal controller for each of the resultant isolated subsystems [14], [15], [16]. Since the effect of interconnections has been neglected in this design procedure, the resultant closed-loop system with these local controllers may be unstable. Even when the interconnected system under the above controllers is stable, the performance index may be poor. As a result, this decentralized control design technique is ineffective in presence of strong coupling between the subsystems.

2. Another approach is to obtain a decentralized static output feedback law by using iterative numerical algorithms in order to minimize the expected value of the quadratic performance index with respect to an initial state with a given probability distribution [17], [18], [19], [20]. This type of design techniques are, in fact, the extended versions of the algorithms for designing optimal centralized static output feedback gain, such as Levine-Athans and Anderson-Moore methods. Although these techniques result in a better performance compared to the preceding method, they have several shortcomings. First of all, they only present necessary conditions, which are mainly in the form of complicated coupled nonlinear matrix equations. Secondly, these iterative algorithms require an initial stabilizing static gain, which should satisfy some requirements. Finally, using a dynamic feedback law instead of a static, the overall performance of the system can be improved significantly.

3. In the third method, the optimal decentralized control problem is formulated by first imposing some assumptions to parameterize all decentralized stabilizing controllers, and then choosing the control parameters such that a desired performance is achieved [21], [22]. However, the resultant equations are either some sophisticated differential matrix equations or some nonconvex relations, which makes them very difficult to solve, in general.

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4. This approach deals with a system with a hierarchical structure. For this class of systems, a rather centralized controller is designed in [23]. The decentralized version of this work is discussed in [24] for a discrete-time system. However, this method can analogously be applied to a continuous-time system, which can be interpreted as follows. In the hierarchical structure, consider the subsystem with the highest-level. Design a centralized local controller for that subsystem assuming that the remaining subsystems are in the open loop, which is desired to account for the performance index. Next, identify the second subsystem with the highest-level, and similarly, design a centralized local controller for it assuming that the first controller designed is a part of the system. Continuing this procedure, one can design local controllers one at a time. The advantage of this method compared to the Method 2, explained above, is that it reduces off-line computation.

However, this approach is inferior to Method 2, because the static gains are computed one at a time in this approach, while in Method 2, explained above, all of the static gains are determined simultaneously. As a result, this approach is proper once the order of the system is so high that the computational complexity is a crucial factor. (note that the first three approaches are for general interconnected systems, while the last one is only for hierarchical systems). There are some other design techniques which are, in fact, combinations of methods 2 and 3 discussed above. All of these approaches are generally incapable of designing a decentralized controller with a satisfactory performance for most systems, including the class of interconnected systems with acyclic structural graphs.

This paper presents a novel design strategy to obtain a high-performance decentralized control law for interconnected systems with leader-follower structure. It will later be shown that the proposed control law outperforms the first, the second and the fourth methods discussed above, and also has a simpler formulation compared to the third method. It is assumed that the state of each subsystem is available in its local output (this is a realistic assumption in many vehicle formation problems, e.g. see [25]), and that a quadratic cost function is defined to evaluate the control performance. The local controller of each subsystem is constructed based on a priori information about the model and initial states of all other subsystems. It is shown that if a priori knowledge of each subsystem is accurate, the performance of the decentralized control system is equal to the minimum achievable performance (which corresponds to the LQR centralized state feedback). In addition, a procedure is proposed to evaluate the closeness of the performance index in the decentralized case to the best achievable performance index (corresponding to the centralized LQR controller) in terms of the amount of inaccuracy in a priori knowledge of any subsystem. This enables the designer to statistically assess the performance of the proposed controller. Moreover, a set of easy-to-check necessary and sufficient conditions for the stability of the decentralized closed-loop system is given for both cases of exact and perturbed models for the system. It is to be noted that providing some information about the model of other subsystems for each individual local controller is performed off-line, in the beginning of control operation, and does not require any communication link between different subsystems. In other words, the proposed control structure is truly decentralized. Optimal cheap control problem is also studied for the leader-follower formation flying. This may require new actuators to be implemented on the vehicles in order to meet a condition on the input structure, which is necessary for the development of the results. While cheap control strategy may not be for many formation flying applications, e.g., constellation of satellites, where it is more desired to apply a minimum fuel control strategy, it can be very useful in certain formation applications involving UAVs with fast tracking missions. Throughout this paper, each vehicle in formation will be referred to as a subsystem and the whole formation consisting of the leader(s) and all followers will be referred to as the system.

II. PROBLEM FORMULATION

Consider a stabilizable interconnected system $S(S_1, S_2, ..., S_\nu)$ with the following state-space equation:
\[
\dot{x} = Ax + Bu \tag{1}
\]
where
\[
A := \begin{bmatrix}
A_{11} & 0 & \cdots & 0 \\
A_{21} & A_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{\nu 1} & A_{\nu 2} & \cdots & A_{\nu \nu}
\end{bmatrix}, \quad B := \begin{bmatrix}
B_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & B_\nu
\end{bmatrix} \tag{2a}
\]
\[
u := \begin{bmatrix}
u_1^T \\
u_2^T \\
\vdots \\
u_\nu^T
\end{bmatrix}^T, \quad x := \begin{bmatrix}
x_1^T \\
x_2^T \\
\vdots \\
x_\nu^T
\end{bmatrix}^T \tag{2b}
\]
and where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $i \in \mathcal{V} := \{1, 2, ..., \nu\}$, are the state and the input of the $i$th subsystem $S_i$, respectively. It is to be noted that the matrices $A$ and $B$ are block lower triangular and block diagonal, respectively. Assume that the state of each subsystem is available in the corresponding local output. In most formation flying applications this is a feasible assumption, and is often met by using GPS for relatively low attitude formations (including UAV missions and many satellite constellation problems) and also sensors mounted on individual vehicles.

**Remark 1:** Consider an interconnected system whose structural graph is acyclic. It is known that the subsystems of this interconnected system can be renumbered in such a way that its corresponding matrix $A$ is lower block diagonal [28]. In
other words, any system with an acyclic structural graph can be converted to a system of the form $\mathcal{S}$ given by (1) and (2), by simply reordering its inputs and outputs properly, if necessary.

Consider now the following quadratic performance index:

$$J = \int_0^\infty (x^TQx + u^TRu) \, dt$$

(3)

where $R \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$ ($n := \sum_{i=1}^{\nu} n_i$, $m := \sum_{i=1}^{\nu} m_i$) are positive definite and positive semi-definite matrices, respectively. For simplicity and without loss of generality, assume that $Q$ and $R$ are symmetric. It is known that if $(A, B)$ is stabilizable, then the performance index (3) is minimized by using the centralized state feedback law:

$$u(t) = -K x(t)$$

(4)

where the gain matrix

$$K := \begin{bmatrix} k_{11} & \cdots & k_{1\nu} \\ \vdots & \ddots & \vdots \\ k_{\nu 1} & \cdots & k_{\nu \nu} \end{bmatrix}, \quad k_{ij} \in \mathbb{R}^{m_i \times n_j}, \ i, j \in \bar{\nu}$$

(5)

is derived from the solution of the Riccati equation [26].

Define the $\nu \times \nu$ block matrix $\Phi = sI - A + BK$ as

$$\Phi := \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1\nu} \\ \vdots & \ddots & \vdots \\ \Phi_{\nu 1} & \cdots & \Phi_{\nu \nu} \end{bmatrix}, \quad \Phi_{ij} \in \mathbb{R}^{n_i \times n_j}, \ i, j \in \bar{\nu}$$

(6)

and for any $i \in \bar{\nu}$, define:

$$M_{1i}(s) := \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1(i-1)} \\ \vdots & \ddots & \vdots \\ \Phi_{(i-1)1} & \cdots & \Phi_{(i-1)(i-1)} \end{bmatrix}, \quad M_{2i} := \begin{bmatrix} \Phi_{1(i+1)} & \cdots & \Phi_{1\nu} \\ \vdots & \ddots & \vdots \\ \Phi_{(i+1)1} & \cdots & \Phi_{(i+1)(i-1)} \end{bmatrix},$$

$$M_{3i} := \begin{bmatrix} \Phi_{1i} & \cdots & \Phi_{1(i-1)} \\ \vdots & \ddots & \vdots \\ \Phi_{\nu 1} & \cdots & \Phi_{\nu (i-1)} \end{bmatrix}, \quad M_{4i}(s) := \begin{bmatrix} \Phi_{1(i+1)} & \cdots & \Phi_{1\nu} \\ \vdots & \ddots & \vdots \\ \Phi_{(i+1)1} & \cdots & \Phi_{(i+1)(i-1)} \end{bmatrix}$$

(7)

It is to be noted that the entries of the matrices $M_{1i}(s)$ and $M_{4i}(s)$ are functions of $s$, but the entries of the two other matrices are constant and independent of $s$. Consider now the following $\nu$ local controllers for the system (1):

$$U_i(s) = \begin{bmatrix} k_{1i} & \cdots & k_{i(i-1)} & k_{ii(i+1)} & \cdots & k_{i\nu} \end{bmatrix} \begin{bmatrix} M_{1i}(s) & M_{2i} \\ M_{3i} & M_{4i}(s) \end{bmatrix}^{-1} \begin{bmatrix} B_1k_{1i} \\ \vdots \\ -A_{vi} + B_{vi} k_{vi} \end{bmatrix} X_i(s)$$

$$= \begin{bmatrix} x^{1,i}_0 \\ \vdots \\ x^{i-1,i}_0 \end{bmatrix} - k_{ii} X_i(s), \ i \in \bar{\nu}$$

(8)

**Theorem 1**: By choosing $x^{ji}_0 = x_j(0), \ i, j \in \bar{\nu}, \ i \neq j$ in (8), the resultant decentralized control law will be equivalent to the optimal centralized controller (4).

**Proof**: Substitute (2), (4), and (5) into (1), take the Laplace transform of the resultant matrix equation, and eliminate its $i$’th row. Rearrange the equation to obtain a relation between $\begin{bmatrix} X_1(s)^T & X_2(s)^T & \cdots & X_{i-1}(s)^T & X_i(s)^T & \cdots & X_{\nu}(s)^T \end{bmatrix}^T$ and $\begin{bmatrix} x_1(0)^T & \cdots & x_i(0)^T & x_{i+1}(0)^T & \cdots & x_{\nu}(0)^T \end{bmatrix}^T$. The proof follows immediately by substituting the resultant relation into the $i$’th block row of the equation (4) in the Laplace domain. ■

Note that $U_i(s)$ in (8) is expressed in terms of the corresponding local information $X_i(s)$ and some constant values $x^{ji}_0, \ j = 1, \ldots, i - 1, i + 1, \ldots, \nu$, but the parameters of the overall system $A_{ji}, B_i, \ i, j \in \bar{\nu}, \ i \geq j$, are assumed to be
known by each subsystem. This assumption, however, is relaxed in Section V. Note also that the control law given by (8) is time-invariant and incrementally linear due to the constants \( x_{ij}^0 \), \( i, j \in \nu \), \( i \neq j \).

Since the control law given by (8) depends on the constant values \( x_{ij}^0 \), \( i, j \in \nu \), \( j \neq i \), it is very important to check the stability of the system with the resultant decentralized controller.

III. STABILITY ANALYSIS VIA GRAPH DECOMPOSITION

It is desired now to find some conditions for the stability of the system (1) when the local controllers (8) are applied to the corresponding subsystems. The following definition will be used in Theorem 2.

**Definition 1:** Consider the system \( S \) given by (1). The modified system \( S^i \), \( i \in \{2, \ldots, \nu\} \), is defined to be a system obtained by removing all interconnections going to the \( i \)'th subsystem in \( S \). The state equation of the modified system \( S^i \) is as follows:

\[
\dot{x} = \tilde{A}^i x + Bu
\]

where \( \tilde{A}^i \) is derived from \( A \) by replacing the first \( i - 1 \) block entries of its \( i \)'th block row with zeros.

**Theorem 2:** Consider the system \( S \) given by (1). Assume that the \( \nu \) local controllers given by (8) are applied to the corresponding subsystems. A sufficient and almost always necessary condition for stability of the resultant decentralized closed-loop system, regardless of the constant values \( x_{ij}^0 \), \( i, j \in \nu \), \( i \neq j \), is that all modified systems \( S^i \), \( i = 2, \ldots, \nu \), are stable under the centralized state feedback law (4).

**Remark 2:** "Almost always necessary" in Theorem 2 means that for the given matrices \( A \) and \( B \), the set of stabilizing gains \( K \) for which the stability of \( S^i \), \( i = 2, \ldots, \nu \) under the centralized state feedback law (4) is violated but the proposed decentralized closed-loop system is still stable, is either an empty set or a hypersurface in the parameter space of \( K \) [27].

**Proof of Theorem 2:**

**Proof of sufficiency:** Suppose that the centralized LTI control system obtained by applying the state feedback law \( u(t) = -K x(t) \) to the modified system \( S^i \) is stable for all \( i \in \{2, ..., \nu\} \). It will be proved by using strong induction that the states of the decentralized control system with the \( \nu \) local controllers (8) are bounded.

**Basis of induction \((i = 1)\):** It is desired to show that the state of the first subsystem is bounded. However, the proof is omitted due to its similarity to the proof of the induction step, which will follow.

**Induction hypothesis:** Suppose that the state of the \( i \)'th subsystem is bounded for \( i = 1, 2, ..., m - 1 \).

**Induction step:** It is required to prove that the state of the \( m \)'th subsystem is bounded. To simplify the formulation, define the following matrices \((m \in \nu)\):

\[
Y_1 := \begin{bmatrix} B_{m1}k_{m1} & \cdots & B_{mm}k_{mm(m-1)} \end{bmatrix}, \quad Y_2 := \begin{bmatrix} B_{m(m+1)}k_{m(m+1)} & \cdots & B_{mm}k_{mm} \end{bmatrix}
\]

\[
Z_1 := \begin{bmatrix} x_0^1 & \cdots & x_{m-1}^m \end{bmatrix}, \quad Z_2 := \begin{bmatrix} -A_{mm} + B_{mm}k_{mm} \end{bmatrix}
\]

\[
A := \begin{bmatrix} A_{m1} & \cdots & A_{mm} \end{bmatrix} := \begin{bmatrix} B_{m1}k_{m1} & \cdots & B_{mm}k_{mm(m-1)} \end{bmatrix}, \quad B := \begin{bmatrix} H_m(s) := [sI - A_{mm} + B_{mm}k_{mm}] \end{bmatrix}
\]

(9a)

(9b)

(9c)

(9d)

Now, by using equations (8) and (1) (with the matrices \( A \) and \( B \) given by (2a)), the following can be concluded:

\[
sX_m(s) = A_{m1}X_1(s) + A_{m2}X_2(s) + \cdots + A_{mm}X_m(s) - B_{mm}k_{mm}X_m(s) - \begin{bmatrix} Y_1 \ Y_2 \end{bmatrix} \begin{bmatrix} M_{1m} & M_{2m} \ M_{3m} & M_{4m}(s) \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \ Z_2 \end{bmatrix} X_m(s) + x_m(0)
\]

Based on the induction assumption, \( x_j(t) \)'s are bounded for \( j = 1, 2, ..., m - 1 \), and consequently they can be considered as exponentially decaying disturbances for the \( m \)'th subsystem. Hence, they do not influence the stability of the \( m \)'th subsystem. Define the homogenous solution \( x_{hm}(t) \) to be the part of the solution for \( x_m(t) \) which corresponds to \( x_1(t) =
\[ \cdots = x_{m-1}(t) = 0. \] This solution satisfies the following equation:

\[
sX_{hm}(s) = A_{mm}X_{hm}(s) - B_mb_{mm}X_{hm}(s) - \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} x_0^m + \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} \begin{bmatrix} Z_{1m} \\ Z_{2m} \end{bmatrix} X_{hm}(s) + x_m(0)
\]

or equivalently

\[
\left( sI - A_{mm} + B_m b_{mm} \right) - \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} \begin{bmatrix} Z_{1m} \\ Z_{2m} \end{bmatrix} X_{hm}(s) = x_m(0) - \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} x_0^m
\]

It can be concluded from (12) that \( x_{hm}(t) \) can be expressed as \( \sum_{i=1}^{l} (p_i(t)x_m(0) + q_i(t)x_m^m) e^{s_i t} \), where \( s = s_i, i = 1, 2, ..., l \), are the roots of the following equation

\[
det \left( H_m(s) - \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} \begin{bmatrix} Z_{1m} \\ Z_{2m} \end{bmatrix} \right) = 0
\]

and also \( p_i(t) \) and \( q_i(t), i = 1, 2, ..., l \) are matrices with polynomial entries of degree less than or equal to the multiplicity of \( s = s_i \) as the root of the above equation, minus one. On the other hand, it can be shown that:

\[
det \begin{bmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{bmatrix} = \det \begin{bmatrix} L_1 & L_3 \\ L_7 & L_9 \end{bmatrix} \times \det \begin{bmatrix} L_5 - \begin{bmatrix} L_4 & L_6 \end{bmatrix} \begin{bmatrix} L_1 & L_3 \\ L_7 & L_9 \end{bmatrix}^{-1} \begin{bmatrix} L_2 \\ L_8 \end{bmatrix} \end{bmatrix}
\]

where \( L_1, L_5, \) and \( L_9 \) are square matrices and \( L_1, L_3, L_7, \) and \( L_9 \) are matrices with the property that \( \begin{bmatrix} L_1 & L_3 \\ L_7 & L_9 \end{bmatrix} \) is nonsingular. Thus, the equation (13) can be simplified as follows:

\[
det \begin{bmatrix} M_{1m}(s) & Z_{1m} \\ Y_{1m} & H_m(s) \\ M_{3m} & Z_{2m} \end{bmatrix} = 0
\]

By substituting the entries of the above matrix from (7) and (9), it can be rewritten in the following simplified form:

\[
det(sI - \tilde{A}^m + BK) = 0
\]

On the other hand, the modes of the closed-loop system \( S^m \) under the feedback law (4) can be obtained from (15). Since it has been assumed that this closed-loop system is stable, all complex numbers \( s_1, ..., s_l \) will be in the open left-half \( s \)-plane. As a result, the state of the \( m \)’th subsystem is bounded.

**Proof of necessity for almost all \( K \)'s:** Suppose that some of the modified systems \( S^2, S^3, ..., S^m \) are not stabilized by the feedback law (4). It is desired to show that the system \( S \) under the proposed local controllers (8) is almost always unstable. Let the first modified system which is unstable under the feedback law (4) be denoted by \( S^m \), i.e. all of the systems \( S^2, S^3, ..., S^{m-1} \) are stabilized by (4). Using the first \( m - 1 \) steps of the induction introduced in the proof of sufficiency, it can be concluded that the states of the subsystems \( 1, 2, ..., m - 1 \) of the system \( S \) under the proposed local controllers (8) are bounded. Now, if the induction continues one more step, it can be concluded that since \( x_j(t) \) is bounded for \( j = 1, 2, ..., m - 1 \), there exists a particular solution for \( x_m(t) \) which approaches zero as time goes to infinity, and the homogenous part of the solution for \( x_m(t) \) (denoted by \( x_m(t) \), which corresponds to \( x_1(t) = \cdots = x_{m-1}(t) = 0 \) satisfies the equation (11), or equivalently (12). Choose any arbitrary unstable mode of the modified system \( S^m \) under the feedback law (4), and denote it with \( s = \sigma^m \). This mode must satisfy (14) or equivalently (13). As mentioned in the proof of sufficiency, \( x_{hm}(t) \) can be expressed as \( \sum_{i=1}^{l} (p_i(t)x_m(0) + q_i(t)x_m^m) e^{s_i t} \), where \( s = s_i, i = 1, 2, ..., l \) are the roots of the equation (13). However, it is required to determine whether or not \( s = \sigma^m \) satisfying (13) appears among \( s = s_i, i = 1, 2, ..., l \). It can be easily verified that \( \sigma^m \neq s_i \), for all \( i = 1, 2, ..., l \) iff both of the following conditions hold.

* \( s = \sigma^m \) is a root of the following equation:

\[
det \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m} & M_{4m}(s) \end{bmatrix} = 0
\]

Note that if the above equation is not satisfied for \( s = \sigma^m \), then

\[
det \left( H_m(s) - \begin{bmatrix} Y_{1m} & Y_{2m} \end{bmatrix} \begin{bmatrix} M_{1m}(s) & M_{2m} \\ M_{3m}(s) & M_{4m}(s) \end{bmatrix}^{-1} \begin{bmatrix} Z_{1m} \\ Z_{2m} \end{bmatrix} \right) = 0
\]
for \( s = \sigma^m \). This will generate a term \( p_i(t)x_m(t)e^{\sigma^mt} \) in \( x_{hm}(t) \) that makes \( x_m(t) \) go to infinity as time increases. Since the matrix in the left side of (16) has been derived from \( sI - A + BK \) by eliminating its \( m \)'th block row and \( m \)'th block column, this requirement is equivalent to the following statement:

The modified system \( S^m \) has an unstable mode \( s = \sigma^m \) under the feedback law (4), and that mode is also the unstable mode of the system \( S \) under the feedback law (4) after isolating its \( m \)'th subsystem (eliminating all of the inputs, outputs, and interconnections of its \( m \)'th subsystem).

- The mode \( s = \sigma^m \) is cancelled out in the following expression:
  
  \[
  \begin{bmatrix}
  Y_{1m} & Y_{2m}
  \end{bmatrix}
  \begin{bmatrix}
  M_{1m}(s) & M_{2m}
  
  \end{bmatrix}^{-1}
  \]

  This means that \( s = \sigma^m \) does not appear in any of the denominators of the entries of the above matrix. Let the matrix obtained from \( A - BK \) by eliminating its \( m \)'th block row and \( m \)'th block column be denoted by \( \Phi^m \). It is easy to verify that this condition is equivalent to the following statement:

  The mode \( s = \sigma^m \) is an unobservable mode of the pair \(([Y_{1m}, Y_{2m}], \Phi^m)\).

  Apparently, for the given matrices \( A \) and \( B \), the set of stabilizing gains \( K \) for which both of the above conditions hold is either an empty set or a hypersurface in the parameter space of \( K \) (for definition of a hypersurface and some similar examples see [27]). If the stabilizing gains \( K \) located on a hypersurface are neglected, \( s = \sigma^m \) appears among \( s = s_i, \ i = 1, 2, ..., l \), which makes \( x_{hm}(t) \) go to infinity, as \( t \) increases. This yields the instability of the decentralized closed-loop control systems.

Theorem 2 states that the stability of the interconnected system given by (1) under the proposed decentralized control law is almost always equivalent to the stability of a set of \( \nu - 1 \) centralized LTI control systems, which can be easily verified from the location of the corresponding eigenvalues.

IV. ROBUST STABILITY ANALYSIS

Since the decentralized control law (8) has been obtained based on the nominal parameters of the system \( S \), it may become unstable once the proposed control law is applied to the perturbed system. Thus, the robust stability of the controller with respect to uncertainties in the original system is an important issue which will be addressed in this section.

Suppose that the decentralized control law (8), which is computed in terms of the nominal parameters \( A \) and \( B \) of the system \( S \), is applied to the system \( \bar{S} \), which is the perturbed version of \( S \) described as follows:

\[
\dot{x} = \bar{A}x + \bar{B}u
\]

where

\[
\bar{A} := \begin{bmatrix}
\bar{A}_{11} & 0 & \cdots & 0 \\
\bar{A}_{21} & \bar{A}_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\bar{A}_{\nu 1} & \bar{A}_{\nu 2} & \cdots & \bar{A}_{\nu \nu}
\end{bmatrix},
\bar{B} := \begin{bmatrix}
\bar{B}_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \bar{B}_\nu
\end{bmatrix}
\]

It is to be noted that the perturbed matrices \( A \) and \( B \) are also block lower triangular and block diagonal, respectively. In other words, it is assumed that parameter variations will not generate new interconnections, i.e. the structural graph of the perturbed system will also be acyclic.

**Definition 2:** The perturbed modified system \( \bar{S}^i, i \in \bar{\nu} \), is defined by:

\[
\dot{x} = \bar{A}^ix + \bar{B}^iu
\]

where the matrix \( \bar{A}^i \) is the same as \( A \), except for its \( i-1 \) block entries \( A_{i1}, ..., A_{i(i-1)} \), which are replaced by zeros, and its \( A_{ii} \) block entry which is replaced by \( \bar{A}_{ii} \). Also, the matrix \( \bar{B}^i \) is the same as \( B \), except for its \((i,i)\) block entry \( B_{ii} \), which is replaced by \( \bar{B}_{ii} \). \( \bar{S}^i \) is, in fact, obtained by modifying \( S \) as follows:

- All interconnections going to the \( i \)'th subsystem are removed.
- The nominal parameters \((A_{ii}, B_{ii})\) of the \( i \)'th subsystem are replaced by the perturbed parameters \((\bar{A}_{ii}, \bar{B}_{ii})\).

**Theorem 3:** Consider the system \( \bar{S} \) given by (17), and assume that the \( \nu \) local controllers given by (8) are applied to the corresponding subsystems. A sufficient and almost always necessary condition for stability of the resultant decentralized closed-loop system, regardless of the constant values \( x_{ij}^{\bar{S}^i} \), \( i, j \in \bar{\nu}, \ i \neq j \), is that all perturbed modified systems \( \bar{S}^i, i = 1, \ldots, \nu \), are stable under the centralized state feedback law (4).

**Proof:** The proof is omitted due to its similarity to the proof of Theorem 2.

**Remark 3:** Since none of the perturbed parameters \( \bar{A}_{ij}, \ i, j \in \bar{\nu}, \ i \neq j \), appear in the perturbed modified systems \( \bar{S}^1, \bar{S}^2, \ldots, \bar{S}^\nu \), the robust stability of the proposed decentralized feedback law is independent of the perturbation of the interconnection parameters (note that this statement is valid for any decentralized control law designed by any arbitrary approach, which is applied to an acyclic interconnected system. In other words, the controller need not be optimal).
Define the perturbation matrix as the perturbed matrix minus the original matrix. The perturbed and perturbation matrices for a matrix $M$ are denoted by $\bar{M}$ and $\Delta M$, respectively. Suppose that the decentralized feedback law (8) is designed in terms of the nominal matrices $A$ and $B$, and then applied to the perturbed system $\bar{S}$ with the state-space matrices $\bar{A}$ and $\bar{B}$. The objective is to find the allowable perturbation matrices $\Delta A = \bar{A} - A$ and $\Delta B = \bar{B} - B$, for which the decentralized closed-loop system will still remain stable. In Theorem 3, a sufficient condition to achieve this objective is presented, which is almost always necessary. Robustness analysis with respect to the perturbation in the parameters of the system can then be summarized as follows:

- For decentralized case, the location of the eigenvalues of the $\nu$ matrices $\bar{A}^i - \bar{B}^i K, \bar{A}^2 - \bar{B}^2 K, ..., \bar{A}^\nu - \bar{B}^\nu K$ should be checked.
- For centralized case, the location of the eigenvalues of the matrix $\bar{A} - \bar{B}K$ should be checked.

Robustness analysis in both classes addresses the following problem, in general:

Consider a Hurwitz matrix $M$, and assume that its entries are subject to perturbations. It is desired to know how much sensitivity the eigenvalues of $M$ are to the variation of its entries. More specifically, it is desired to find out how much the matrix $M$ can be perturbed so that the resultant matrix is still Hurwitz.

This problem has been addressed in the literature using different mathematical approaches [29], [30], [31]. Sensitivity of the eigenvalues to the variation of its entries depends, in general, on several factors such as the norm of the perturbation matrix, structure of the matrix (represented by condition number or eigenvalue condition number [30]), and repetition or distinction of the eigenvalues.

**Theorem 4:** The bound on the Frobenius norm of the perturbation matrix corresponding to each of the matrices $\bar{A}^i - \bar{B}^i K$, $i = 1, 2, ..., \nu$ in the decentralized case is less than or equal to that of the perturbation matrix corresponding to $\bar{A} - \bar{B}K$ in the centralized case.

**Proof:** The following relation holds for the decentralized case:

$$
\| \Delta (\bar{A}^i - \bar{B}^i K) \|_F = \| (\bar{A}^i - \bar{B}^i K) - (\bar{A}^i - BK) \|_F = \sqrt{\sum_{j=1}^{\nu} \left( \| \Delta B_i k_{ij} \|_F^2 + \| \Delta A_{ii} \|_F^2 \right)}
$$

This results in:

$$
\| \Delta (\bar{A}^i - \bar{B}^i K) \|_F \leq \sqrt{\Gamma_{\text{dec}_i}}, \quad i \in \nu
$$

where

$$
\Gamma_{\text{dec}_i} := \sum_{j=1}^{\nu} \left( \| \Delta B_i k_{ij} \|_F^2 + \| \Delta A_{ii} \|_F^2 \right), \quad i \in \nu
$$

For the centralized case, on the other hand, one can write

$$
\| \Delta (\bar{A} - \bar{B}K) \|_F = \| (\bar{A} - \bar{B}K) - (A - BK) \|_F = \sqrt{\sum_{i=1}^{\nu} \sum_{j=1}^{i} \| \Delta B_i k_{ij} \|_F^2 + \sum_{i=1}^{\nu} \sum_{j=i+1}^{\nu} \| \Delta A_{ij} \|_F^2}
$$

Thus,

$$
\| \Delta (\bar{A} - \bar{B}K) \|_F \leq \sqrt{\Gamma_{\text{cen}}}
$$

where

$$
\Gamma_{\text{cen}} := \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \| \Delta B_i k_{ij} \|_F^2 + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \| \Delta A_{ij} \|_F^2
$$

It is apparent from (19) and (21), that

$$
\Gamma_{\text{dec}_1} + \Gamma_{\text{dec}_2} + \cdots + \Gamma_{\text{dec}_\nu} \leq \Gamma_{\text{cen}}
$$

Therefore,

$$
\sqrt{\Gamma_{\text{dec}_i}} \leq \sqrt{\Gamma_{\text{cen}}}, \quad i = 1, 2, ..., \nu
$$

The proof follows immediately from (18), (20) and (23). It is to be noted that the inequality (23) obtained above, is more conservative than (22), obtained in the preceding step.

According to Theorem 4, there are $\nu$ perturbed matrices in the decentralized case, and the bound on the Frobenius norm of the perturbation matrix for each of them is less than or equal to the bound on the Frobenius norm of the corresponding perturbation matrix in the centralized case. Therefore, it can be concluded from the above discussion and Remark 3, that the proposed decentralized controller is expected to perform better than the centralized counterpart in terms of robust stability.
with respect to the parameter variations of the system. This result can also be deduced intuitively, because for any subsystems $i$ and $j$ ($i > j$):
- In the centralized case, any perturbation in subsystem $i$ will influence the state of subsystem $j$ through the feedback and can cause the instability of the closed-loop system.
- In the decentralized case, no perturbation in subsystem $i$ can influence the state of subsystem $j$ through the feedback or through the interconnections, due to the particular structure of the system (i.e., lower-triangular structure of $A$ and diagonal structure of $B$).

V. NON-IDENTICAL LOCAL BELIEFS ABOUT THE SYSTEM MODEL

In practice, different local controllers may assume different models for the overall system. It is desired now to find some results similar to the ones presented in Theorem 3, under the above condition.

Suppose that control agent $l$ assumes the matrices $\hat{A}^l$ and $\hat{B}^l$ instead of the matrices $A$ and $B$ in the state-space representation (1) of the system $S$. Denote the $(i, j)$ block entry of $\hat{A}^l$ with $\hat{A}^l_{ij} \in \mathbb{R}^{n_i \times n_j}$, for any $i, j \in \bar{\nu}$, $i \geq j$, and the $(i, i)$ block entry of $\hat{B}^l$ with $\hat{B}^l_{i} \in \mathbb{R}^{n_i \times m_l}$ for any $i \in \bar{\nu}$. Now, for any $l \in \bar{\nu}$, replace $A$ and $B$ in the equation (1) with $\hat{A}^l$ and $\hat{B}^l$, respectively. Then solve the corresponding LQR problem for the above matrices, to obtain the optimal static gain $\hat{K}^l$, whose $(i, j)$ block entry is denoted by $\hat{k}^l_{ij}$, for all $i, j \in \bar{\nu}$. Define the matrices $\hat{M}_1^l(s), \hat{M}_2^l, \hat{M}_3^l$, and $\hat{M}_4^l(s)$ similarly to the matrices in (7), by replacing $\Phi = sI - A + BK$ in (6) with $\Phi^l := sI - \hat{A}^l + \hat{B}^l\hat{K}^l$. Therefore, the $l$'th local control law, in this case, is given by ($l \in \bar{\nu}$):

$$U_l(s) = \left[ \hat{k}^l_{11} \ldots \hat{k}^l_{i(l-1)} \hat{k}^l_{i(l+1)} \ldots \hat{k}^l_{i\nu} \right] \left[ \begin{array}{cc} \hat{M}_1^l(s) & \hat{M}_2^l \\ \hat{M}_3^l & \hat{M}_4^l(s) \end{array} \right]^{-1} \left[ \begin{array}{c} \hat{B}^l_{i} \hat{k}^l_{i1} \\ \vdots \\ -\hat{A}^l_{i(l+1)i} + \hat{B}^l_{i} \hat{k}^l_{i(l+1)i} \\ \vdots \\ -\hat{A}^l_{i\nu l} + \hat{B}^l_{i} \hat{k}^l_{i\nu l} \end{array} \right] X_l(s)$$

(24)

Definition 3: The uncertain model $\bar{S}^l$, $l \in \bar{\nu}$, is defined by:

$$\dot{x}(t) = A^l x(t) + B^l u(t)$$

where the matrix $A^l$ is the same as $\hat{A}^l$, except for its $l - 1$ block entries $\hat{A}^l_{11}, \ldots, \hat{A}^l_{i(l-1)}$, which are replaced by zeros, and its $\hat{A}^l_{ii}$ block entry, which is replaced by $A_{ii}$. Also, the matrix $B^l$ is the same as $\hat{B}^l$, except for its $(l, l)$ block entry $\hat{B}^l_{ll}$, which is replaced by $B_{ll}$.

Corollary 1: Consider the system $\bar{S}$ given by (17). Assume that the $\nu$ local controllers given by (24) are applied to the corresponding subsystems. A sufficient and almost always necessary condition for stability of the resultant decentralized closed-loop system, regardless of the constant values $x_0^{l,j}$, $i, j \in \bar{\nu}$, $i \neq j$, is that the uncertain system $S^l$ is stable under the centralized state feedback law $u(t) = -\hat{K}^l x(t)$, for all $i \in \bar{\nu}$.

Proof: The proof is omitted due to its similarity to the proof of Theorem 2. $

VI. CENTRALIZED AND DECENTRALIZED PERFORMANCE COMPARISON

So far, a decentralized control law has been proposed for a class of stabilizable LTI systems with the property that if the modeling parameters and the initial state of each subsystem are available in all other subsystems, then the proposed controller will be equivalent to the optimal centralized controller. It is to be noted that the equalities $\hat{A}^l = A^l, \hat{B}^l = B^l, l \in \bar{\nu}$, will hereafter be assumed to simplify the presentation of the properties of the decentralized control proposed in this paper. Note that the results presented under this assumption, can simply be extended to the general case. It has also been shown that if the conditions of Theorem 2 are satisfied, then by using any arbitrary constant values instead of the initial states of other subsystems form each subsystem’s view, the resultant decentralized closed-loop system will remain stable,
which implies that the deviation $\Delta J$ of the resultant quadratic performance index (3) from the optimal performance index corresponding to the centralized LQR controller remains finite. The following definitions are used to find $\Delta J$.

**Definition 4:** Define $\Delta x_j(t), i, j \in \mathcal{V}, i \neq j$, as the difference between the initial state of the $j$'th subsystem $x_j(0)$ and $x_j^{i,j}$. Throughout the remainder of the paper, this difference will be referred to as the prediction error of the initial state.

Due to the prediction errors defined above, there will be a deviation in the state $x_i(t)$ and control input $u_i(t), i \in \mathcal{V}$, of the resultant decentralized control system compared to those of the centralized counterpart. Denote the state and the control input deviations with $\Delta x_i(t)$ and $\Delta u_i(t)$, respectively.

**Definition 5:** The matrices $\Delta x_0$ and $\Delta m x(0)$, $m \in \mathcal{V}$, are defined as follows:

$$\Delta x_0 = \begin{bmatrix} \Delta_1 x(0) \\ \Delta_2 x(0) \\ \vdots \\ \Delta_n x(0) \end{bmatrix}, \quad \Delta m x(0) = \begin{bmatrix} \Delta_m x_1(0) \\ \vdots \\ \Delta_m x_{i-1}(0) \\ \Delta_m x_i(0) \\ \vdots \\ \Delta_m x_{\nu}(0) \end{bmatrix}, \quad m = 1, 2, ..., \nu$$

The following algorithm is presented to find $\Delta J$ in terms of the prediction errors $\Delta x_j(t), i, j \in \mathcal{V}, i \neq j$.

**Algorithm 1:**

1) Find $\Delta X_1(s)$ in terms of $\Delta_1 x(0)$ by using equation (10) (for $m = 1$), which can be expressed as $\Delta X_1(s) = F_{11}(s)\Delta_1 x(0)$. Substitute $\Delta X_1(s)$ into equation (8) for $i = 1$ to obtain $\Delta U_1(s)$ only in terms of $\Delta_1 x(0)$, i.e. $\Delta U_1(s) = G_{11}(s)\Delta_1 x(0)$.

2) $\Delta X_m(s)$ and $\Delta U_m(s)$ have been computed for $i = 1, 2, ..., m - 1$ in terms of prediction errors in the previous steps of the algorithm. Now, for $i = m$, substitute the expressions obtained for $\Delta X_1(s), \Delta X_2(s), ..., \Delta X_{m-1}(s)$ into equation (10) to find an equation relating $\Delta X_m(s)$ to the prediction errors. Let this equation be represented by $\Delta X_m(s) = F_{m1}(s)\Delta_1 x(0) + F_{m2}(s)\Delta_2 x(0) + \cdots + F_{mm}(s)\Delta_m x(0)$. By substituting this expression into (8) for $i = m$, $\Delta U_m(s)$ will be found in terms of the prediction errors, i.e. $\Delta U_m(s) = G_{m1}(s)\Delta_1 x(0) + G_{m2}(s)\Delta_2 x(0) + \cdots + G_{mm}(s)\Delta_m x(0)$.

The algorithm continues up to step $\nu$. It is obvious from the expressions in step $m$ of Algorithm 1, that the deviation in the state of each subsystem depends not only on its own prediction errors, but also on the prediction errors of all previous subsystems due to the interconnections. The results obtained from the algorithm can be written in the matrix form as follows:

$$\Delta X(s) = F(s)\Delta x_0, \quad \Delta U(s) = G(s)\Delta x_0$$

where

$$F(s) = \begin{bmatrix} F_{11}(s) & 0 & 0 & \cdots & 0 \\ F_{21}(s) & F_{22}(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{\nu 1}(s) & F_{\nu 2}(s) & F_{\nu 3}(s) & \cdots & F_{\nu \nu}(s) \end{bmatrix}, \quad \Delta X(s) = \begin{bmatrix} \Delta X_1(s) \\ \vdots \\ \Delta X_\nu(s) \end{bmatrix}$$

$$G(s) = \begin{bmatrix} G_{11}(s) & 0 & 0 & \cdots & 0 \\ G_{21}(s) & G_{22}(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{\nu 1}(s) & G_{\nu 2}(s) & G_{\nu 3}(s) & \cdots & G_{\nu \nu}(s) \end{bmatrix}$$

Therefore, the deviation of the performance index due to the prediction errors can be obtained as follows:

$$\Delta J = \int_0^\infty \left( [x + \Delta x]^T Q [x + \Delta x] + [u + \Delta u]^T R [u + \Delta u] \right) dt - \int_0^\infty (x^T R x + u^T R u) dt$$

$$= \int_0^\infty (x^T Q \Delta x + \Delta x^T Q x + \Delta x^T Q \Delta x + u^T R \Delta u + \Delta u^T R \Delta u + \Delta u^T R \Delta u) dt$$

(26)

It is to be noted that $x$ and $u$ are the state and the input of the centralized closed-loop system, and $x + \Delta x$ and $u + \Delta u$ are those of the decentralized closed-loop system. On the other hand, equations (1) and (4) yield:

$$X(s) = W(s)x(0), \quad U(s) = Z(s)u(0)$$
where
\[ W(s) = (SI - A + BK)^{-1}, \quad Z(s) = -K(SI - A + BK)^{-1} \]

Suppose that \( w(t), z(t), f(t) \) and \( q(t) \) represent the time domain functions corresponding to \( W(s), Z(s), F(s) \) and \( G(s) \), respectively. Substituting these time functions into (26) results in:

\[
\Delta J = \int_0^\infty (x(0)^T w(t)^T Q f(t) dx_0 + \Delta x_0^T f(t)^T Q w(t) x(0) + \Delta x_0^T f(t)^T Q f(t) \Delta x_0) dt
\]
\[
+ \int_0^\infty (x(0)^T z(t)^T R g(t) \Delta x_0 + \Delta x_0^T g(t)^T R z(t) x(0) + \Delta x_0^T g(t)^T R g(t) \Delta x_0) dt
\]

Due to the causality of the system, the arguments of both integrals in the above equation are zero for negative time. As a result, the interval for both integrals can be changed from \((0, +\infty)\) to \((-\infty, +\infty)\). Hence, one can use the Parseval’s formula to obtain:

\[
\Delta J = \frac{1}{2\pi} \int_{-\infty}^{\infty} (x(0)^T W(j\omega)^T Q F(-j\omega) \Delta x_0 + \Delta x_0^T F(j\omega)^T Q W(-j\omega) x(0) + \Delta x_0^T F(j\omega)^T Q F(-j\omega) \Delta x_0) d\omega
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} (x(0)^T Z(j\omega)^T R G(-j\omega) \Delta x_0 + \Delta x_0^T G(j\omega)^T R Z(-j\omega) x(0) + \Delta x_0^T G(j\omega)^T R G(-j\omega) \Delta x_0) d\omega
\]

Define the following matrices:

\[
V_{12} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (W(j\omega)^T Q F(-j\omega) + Z(j\omega)^T R G(-j\omega)) d\omega
\]

\[
V_{21} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (F(j\omega)^T Q W(-j\omega) + G(j\omega)^T R Z(-j\omega)) d\omega
\]

\[
V_{22} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (F(j\omega)^T Q F(-j\omega) + G(j\omega)^T R G(-j\omega)) d\omega
\]

It is to be noted that since \( R \) and \( Q \) are assumed to be symmetric matrices, \( V_{21} \) and \( V_{22} \) are equal to \( V_{12}^T \) and \( V_{22}^T \), respectively. Thus, it can be concluded from (27) that:

\[
\Delta J = x(0)^T V_{12} \Delta x_0 + \Delta x_0^T V_{21} x(0) + \Delta x_0^T V_{22} \Delta x_0
\]
\[
\quad = \begin{bmatrix} x(0)^T & \Delta x_0^T \end{bmatrix} \begin{bmatrix} 0 & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} x(0) \\ \Delta x_0 \end{bmatrix} = \begin{bmatrix} x(0)^T & \Delta x_0^T \end{bmatrix} \begin{bmatrix} 0 & V_{12}^T \\ V_{21}^T & V_{22}^T \end{bmatrix} \begin{bmatrix} x(0) \\ \Delta x_0 \end{bmatrix}
\]

Proposition 1: The performance deviation \( \Delta J \) can be written as:

\[
\Delta J = \Delta x_0^T V_{22} \Delta x_0
\]

Proof: Consider an arbitrary \( x(0) \), and assume that \( \Delta x_0 \) is a variable vector. Note that the entries of \( \Delta x_0 \) can take any values, because they represent prediction errors of the initial states. \( \Delta J \) given in (29) has the following properties:

- \( \Delta J \) is always nonnegative, because the centralized optimal performance index has the smallest value among all performance indices resulted by using any type of centralized or decentralized controller.
- Substituting \( \Delta x_0 = 0 \) in (29) yields \( \Delta J = 0 \).
- \( \Delta J \) is continuous with respect to each of the entries of the variable \( \Delta x_0 \), because \( \Delta J \) is quadratic. It can be concluded from the above properties that \( \Delta x_0 = 0 \) is an extremum point for \( \Delta J \). Thus, the partial derivative of \( \Delta J \) with respect to \( \Delta x_0 \) is equal to zero at \( \Delta x_0 = 0 \). Hence:

\[
\left| \begin{bmatrix} x(0)^T V_{12} + (V_{12}^T x(0))^T & \Delta x_0^T \left( V_{22}^T + V_{22} \right) \end{bmatrix} \right|_{\Delta x_0=0} = 0
\]

which results in \( x(0)^T V_{12} = 0 \). This implies that any arbitrary vector \( x(0) \) is in the null space of \( V_{12} \), or equivalently \( V_{12} = 0 \). The proof follows immediately from substituting \( V_{12} = 0 \) into (29), and noting that \( V_{21} = V_{12}^T = 0 \).

Remark 4: Equation (30) states that for finding the performance deviation \( \Delta J \), there is no need to obtain the time functions \( w(t) \) and \( z(t) \). In other words, only the functions \( f(t) \) and \( g(t) \) are required for performance evaluation. Furthermore, one can directly use the Laplace transforms \( F(s) \) and \( G(s) \), and substitute \( s = \pm j\omega \) to obtain \( \Delta J \) through (28) and (30).

Remark 5: It can be concluded from (30), that the performance deviation \( \Delta J \) depends only on the prediction error of the initial state, not the initial state itself.

Theorem 5: To minimize the expected value of the performance index \( J \), the constant value \( x_0^{i,j} \) should be chosen equal to the expected value of the \( j \)’th subsystem’s initial state from the \( i \)’th subsystem’s view for any \( i, j \in \nu, i \neq j \).
Proof: Consider an arbitrary initial state $x(0)$. It can be concluded from (25) and Definition 4 that $\Delta x_0$ can be written as $\hat{x}_0 - x_0$, where $\hat{x}_0$ is a vector whose entries are related to the constant values $x_0^{j,m}$. Also, $x_0$ is a vector whose entries are related to the initial states $x_m(0)$, $m \in \nu$. Consequently,

$$E\{\Delta J\} = E\{\Delta x_0^T V_{22} \Delta x_0\} = E\{(\hat{x}_0^T - x_0^T) V_{22} (\hat{x}_0 - x_0)\}$$

$$= \hat{x}_0^T V_{22} \hat{x}_0 - E\{x_0\}^T V_{22} x_0 - \hat{x}_0^T V_{22} E\{x_0\} + E\{x_0\}^T V_{22} E\{x_0\}$$

To minimize the above expression, take its partial derivative with respect to $\hat{x}_0$ and equate it to zero as follows:

$$\hat{x}_0^T (V_{22} + V_{22}^T) - E\{x_0\}^T V_{22} - (V_{22} E\{x_0\})^T = 0$$

which results in:

$$(\hat{x}_0^T - E\{x_0\}) (V_{22} + V_{22}^T) = 0 \quad (31)$$

Since the optimal control strategy is unique, $\Delta J$ should be positive for any nonzero $\Delta x_0$. As a result, the matrix $V_{22}$ in (30) is positive definite and consequently, the matrix $V_{22} + V_{22}^T$ is positive definite as well. Thus, the determinant of the matrix $V_{22} + V_{22}^T$ is nonzero, and so it can be concluded from (31) that $\hat{x}_0^T - E\{x_0\} = 0$, or equivalently $E\{\Delta x_0\} = E\{\hat{x}_0 - x_0\} = 0$. In other words, the expected value of any entry of $\Delta x_0$ should be zero. Thus, it can be deduced from (25) and Definition 4 that

$$E\{x_0^{j,m} - x_j(0)\} = 0, \quad j, m \in \nu, \ j \neq m$$

This relation states that the best choice for $x_0^{j,m}$ is equal to $E_m\{x_j(0)\}$, the expected value of the initial state of the $j$'th subsystem from the $m$'th subsystem’s view.

Remark 6: One can use Proposition 1 and Theorem 5 to obtain statistical results for the performance deviation $\Delta J$ in terms of the expected values of the initial states of the subsystems. This can be achieved by using Chebyshev’s inequality. This enables the designer to determine the maximum allowable standard deviation for $\Delta x_0$ to achieve a performance deviation within a prespecified region with a sufficiently high probability (e.g. 95%).

Remark 7: Suppose that the initial state of an acyclic interconnected system is a random variable whose mean $\bar{x}_0$ and covariance matrix are given. Consider a decentralized control law obtained by using the method in [17] (i.e., the second approach discussed in the introduction). For any given initial state $x(0)$, compute the quadratic performance index (for any given $Q$ and $R$) of the resultant system and denote it with $J_1(x(0))$. Define now $J_2(x(0))$ as the quadratic performance index (with the same parameters $Q$ and $R$) for the closed-loop system with the controller proposed in this paper. It is to be noted that to design this controller, the prediction values used in (8) are replaced by their corresponding mean values, as explained in Theorem 5. Moreover, define $J_c(x(0))$ as the minimum achievable performance index for the centralized case. According to Theorem 1, $J_2(\bar{x}_0) = J_c(\bar{x}_0)$, which implies that $J_2(\bar{x}_0) < J_1(\bar{x}_0)$. This means that there is a region $\mathcal{R}$ around the point $x_0$ in the $n$ dimensional space, such that for any $x(0)$ in this region, the inequality $J_2(x(0)) < J_1(x(0))$ holds. On the other hand, if the function $J_2(x(0))$ is smooth around $\bar{x}_0$, the initial state of the system will have a greater likelihood inside the region $\mathcal{R}$ rather than outside of it, in which case the controller proposed in this paper will outperform the one obtained by the method proposed in [17]. It is to be noted that to evaluate the smoothness of the function $J_2(x(0))$ one can use the formula (30) to obtain the function $J_2(x(0))$, in a quadratic form, while for the numerical method such as the one in [17], there is no closed-form formula for $J_1(x(0))$ in terms of the initial state. Similar comparison can analogously be made between the method presented in this paper and the method given in [14], [15] and [16] (first approach discussed in the introduction).

VII. DECENTRALIZED HIGH-PERFORMANCE CHEAP CONTROL

Consider now the cheap control optimization problem, where it is desired to minimize a quadratic performance index of the following form:

$$J = \int_0^\infty (x^T Q x + \varepsilon u^T R u) \ dt \quad (32)$$

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are positive definite matrices, and $\varepsilon$ is a positive number which is chosen sufficiently close to zero for this type of problem. For simplicity and without loss of generality, assume again that $Q$ and $R$ are symmetric. Consider the matrix $K_\varepsilon$ such that the feedback law:

$$u(t) = -K_\varepsilon x(t) \quad (33)$$

minimizes the performance index (32). In the remainder of this section, assume that $K$ given in (5) is equal to $K_\varepsilon$. According to Theorem 2, the local controllers given by (8) can stabilize the system $\mathcal{S}$, if all modified systems $\mathcal{S}_i^\nu$, $i = 2, 3, ..., \nu$, under the feedback law (33) are stable. These conditions are also almost always necessary. In sequel, it will be shown that if $\text{det} (BR^{-1}B^T) \neq 0$, and if $\varepsilon$ is sufficiently close to zero, there is no need to check the stability of the $\nu - 1$ closed-loop modified systems.
Theorem 6: Assume that $s_1^*, s_2^*, \ldots, s_n^*$ are the eigenvalues of the system $S$ under the feedback law $u(t) = -K_x x(t)$, and that the determinant of the matrix $BR^{-1}B^T$ is nonzero. Then, as $\varepsilon$ approaches zero, $\sqrt{\varepsilon} s_1^*, \sqrt{\varepsilon} s_2^*, \ldots, \sqrt{\varepsilon} s_n^*$ converge to $n$ negative (nonzero) real numbers $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$, which satisfy the following equation:

$$\det (s^2 I - WQ) = 0, \quad i = 1, 2, \ldots, n$$

(34)

where $W = BR^{-1}B^T$.

Proof: It is known that the state and costate of the system $S$ under the optimal feedback law $u(t) = -K_x x(t)$ satisfy the following equation [26]:

$$\begin{bmatrix} \dot{x} \\ \lambda \end{bmatrix} = H \times \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad H = \begin{bmatrix} A & -W \\ -Q & -A^T \end{bmatrix}, \quad W = BR^{-1}B^T$$

The matrix $H$ has $2n$ eigenvalues in mirror-image pairs with respect to the imaginary axis. Those eigenvalues which are in the left-half $s$-plane are the eigenvalues of the closed-loop system under the feedback law $u(t) = -K_x x(t)$. The eigenvalues of the matrix $H$ are obtained from the following equation:

$$\det \left( sI - \frac{BR^{-1}B^T}{s} \right) = 0$$

(35)

Let the roots of the above equation be denoted by $s_1^*, s_2^*, \ldots, s_n^*$, where $s_i^* = -s_i^*$, $Re \{s_i^*\} \leq 0$, for $i = 1, 2, \ldots, n$. One can multiply the first $n$ rows and the last $n$ columns of the matrix in the left side of (35) by $\sqrt{\varepsilon}$ to obtain the following relation:

$$\begin{align*}
\det \left[ sI - A & \frac{W}{sI + A^T} \\ sI & sI + A^T \right] = \frac{1}{\varepsilon^n} \det \left[ s^2 I - \sqrt{\varepsilon}A & W \\ \sqrt{\varepsilon}sI + \sqrt{\varepsilon}A & W \right] \\
\text{Hence, it can be concluded from (35) and (36), that } s_1^*, s_2^*, \ldots, s_n^* \text{ are the roots of the following equation:} \\
\det \left[ sI - \sqrt{\varepsilon}A & W \\ \sqrt{\varepsilon}sI + \sqrt{\varepsilon}A & W \right] = 0
\end{align*}$$

(36)

Define $\hat{s}^*_i = \sqrt{\varepsilon} s_i^*$, $i = 1, 2, \ldots, n$. Consequently, $\hat{s}_1^*, \hat{s}_2^*, \ldots, \hat{s}_n^*$ satisfy the following equation:

$$\begin{align*}
\det \left[ sI - \sqrt{\varepsilon}A & W \\ sI + \sqrt{\varepsilon}A & W \right] = 0
\end{align*}$$

(37)

It can be easily verified that the above equation is equivalent to

$$s^{2n} + p_{2n-1}(\sqrt{\varepsilon})s^{2n-1} + p_{2n-2}^* s^{2n-2} + \cdots + p_1(\sqrt{\varepsilon})s + p_0(\sqrt{\varepsilon}) = 0$$

(38)

where $p_i(\sqrt{\varepsilon})$, $i = 1, 2, \ldots, 2n - 1$, is a polynomial in $\sqrt{\varepsilon}$. Obviously, as $\varepsilon$ approaches zero, $\hat{s}_1^*, \hat{s}_2^*, \ldots, \hat{s}_n^*$ (which satisfy the equation (37) or equivalently (38)), converge to $n$ definite and finite complex numbers denoted by $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$ (note that the roots of a polynomial with finite coefficients are finite), and also they satisfy the equation (38) for $\varepsilon = 0$, i.e.

$$\hat{s}_1^* + p_{2n-1}(0)\hat{s}_1^{2n-1} + p_{2n-2}(0)\hat{s}_1^{2n-2} + \cdots + p_1(0)\hat{s}_1 + p_0(0) = 0, \quad i = 1, 2, \ldots, n$$

To find $\hat{s}_i$, replace $\varepsilon$ with zero and substitute $s = \hat{s}_i$ in the equation (37). This results in:

$$\det \left[ \hat{s}_i I & W \\ \hat{s}_i I & \hat{s}_i I \right] = 0, \quad i = 1, 2, \ldots, n$$

The above equation can be simplified as follows:

$$0 = \det \left[ \hat{s}_i I & W \\ \hat{s}_i I & \hat{s}_i I \right] = \det (\hat{s}_i^2 I - WQ), \quad i = 1, 2, \ldots, n$$

So far, it has been shown that as $\varepsilon$ approaches zero, $\sqrt{\varepsilon} s_1^*, \sqrt{\varepsilon} s_2^*, \ldots, \sqrt{\varepsilon} s_n^*$ converge to the definite numbers $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$, which satisfy equation (34). Since $R$ is positive definite and symmetric, $W$ is positive definite and symmetric as well (note that $\det(W) \neq 0$). Using Cholesky decomposition, one can easily conclude that all of the eigenvalues of the matrix $WQ$ are positive real numbers. Therefore, the equation (34) implies that $\hat{s}_1^*, \hat{s}_2^*, \ldots, \hat{s}_n^*$ are positive real numbers and consequently, $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$ are purely real. Since the feedback law $u(t) = -K_x x(t)$ stabilizes the system $S$, all of the eigenvalues of this closed-loop system are located in the left-half $s$-plane. As a result, $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$ are non-positive. Also, it is apparent that none of $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n$ are zero, because in that case $\det(WQ) = 0$, which is a contradiction to the assumption of positive definite $Q$ (note that $\det(W) \neq 0$).

It is to be noted that the result of Theorem 6 is an extension of the existing results for the modes of optimal closed-loop SISO systems and the corresponding inverse root characteristic equation [32], to the MIMO case.
As an example, consider a system consisting of two 2-input 2-output subsystems and the following state-space matrices:

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 \\
-2 & 30 & 0 & 0 \\
4 & 6 & 1 & 2 \\
-5 & 5 & 7 & 5
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 60 & 0 & 0 \\
2 & 6 & 0 & 0 \\
0 & 0 & 10 & 1 \\
0 & 0 & 3 & 3
\end{bmatrix}
\] (39)

Solving the centralized optimal LQR problem for \( R = Q = I \) and multiplying the eigenvalues of the resultant closed-loop system (under the feedback law (33)) by \( \sqrt{\varepsilon} \) as described in Theorem 6, will result in \( \{ \sqrt{\varepsilon}s_1^*, \sqrt{\varepsilon}s_2^*, \sqrt{\varepsilon}s_3^*, \sqrt{\varepsilon}s_4^* \} \). The following sets of eigenvalues are obtained for \( \varepsilon = 10^{-2}, 10^{-3}, 10^{-4} \) and \( 10^{-5} \), respectively:

\[
\{-60.336, -10.609, -3.4573, -2.7356\}, \quad \{-60.339, -10.608, -2.5601, -2.0257\}, \\
\{-60.339, -10.608, -2.5469, -1.8147\}, \quad \{-60.339, -10.608, -2.5455, -1.7924\}
\] (40)

On the other hand, the roots of (34) are given by:

\[
\{ \pm 60.339, \pm 10.608, \pm 2.5453, \pm 1.7899 \}\] (41)

From (40) and (41), it is evident that as \( \varepsilon \) become smaller, the modes of the optimal closed-loop system under the feedback law (33) approach the negative elements of the set (41), as expected from Theorem 6 (Note that \( BR^{-1}B^T \) is nonsingular in this example).

**Lemma 1:** Consider two arbitrary symmetric positive-definite matrices \( G \) and \( H \). There is a unique positive definite matrix \( X \) which satisfies the following relation:

\[
XGX = H
\]

**Proof:** It is known that every symmetric positive-definite matrix can be uniquely written as the square of another symmetric positive definite matrix. Therefore, there is a unique positive definite matrix \( \hat{G} \) such that \( G = \hat{G}^2 \). Define \( Y = GXG \), or equivalently \( X = \hat{G}^{-1}YG^{-1} \). It is clear that since \( \hat{G} \) and \( X \) are positive definite and \( \hat{G} \) is symmetric, \( Y \) is also positive definite, and

\[
H = X\hat{G}^2X = \hat{G}^{-1}YG^{-1}\hat{G}^2\hat{G}^{-1}Y\hat{G}^{-1} = \hat{G}^{-1}Y^2\hat{G}^{-1}
\]

or equivalently:

\[
Y^2 = \hat{G}HG
\] (42)

Similarly, since \( H \) and \( G \) are positive definite, \( \hat{G}HG \) is positive definite as well. Therefore, there is a unique positive definite matrix \( \hat{Y} \) whose square is equal to \( \hat{G}HG \). The matrix \( \hat{Y} \) satisfies the equation (42), and thus \( X \) is determined to be equal to \( \hat{G}^{-1}Y\hat{G}^{-1} \), which is also unique.

**Theorem 7:** Suppose that the matrix \( W \) corresponding to the system (1) and the performance index (32) is nonsingular. Consider the modified system \( S^j \), \( j \in \{ 2, 3, ..., v \} \). There exists a finite \( \varepsilon^* > 0 \) such that for every positive real number \( \varepsilon \) less than \( \varepsilon^* \), the modified system \( S^j \) is stable under the feedback law (33).

**Proof:** Assume that the modes of the system \( S \) under the feedback law (33) are \( s_1^*, s_2^*, ..., s_n^* \). It is clear that these modes satisfy the following equation:

\[
det(s_i^*I - A + BK_\varepsilon) = 0, \quad i = 1, 2, ..., n
\] (43)

Suppose that \( P_\varepsilon \) is the solution of the Riccati equation for the system \( S \) and the performance index (32). Thus,

\[
-P_\varepsilon A - A^T P_\varepsilon - Q + \frac{1}{\varepsilon} P_\varepsilon BR^{-1}B^T P_\varepsilon = 0
\] (44)

Since \( K_\varepsilon = \frac{1}{\varepsilon} R^{-1}B^T P_\varepsilon \) and \( W = BR^{-1}B^T \), the equation (43) can be rewritten as

\[
det\left(s_i^*I - A + \frac{1}{\varepsilon} WP_\varepsilon\right) = 0, \quad i = 1, 2, ..., n
\] (45)

According to Theorem 6, as \( \varepsilon \) approaches zero, \( \sqrt{\varepsilon}s_i^* \) converges to the negative definite number \( \hat{s}_i \) for \( i = 1, 2, ..., n \). Using this approximation and substituting it into (45) will result in (as \( \varepsilon \to 0 \)):

\[
det\left(\frac{\hat{s}_i}{\sqrt{\varepsilon}}I - A + \frac{1}{\varepsilon} WP_\varepsilon\right) \to 0, \quad i = 1, 2, ..., n
\]

Define \( \hat{P}_\varepsilon := \frac{P_\varepsilon}{\sqrt{\varepsilon}} \). It can then be concluded from the above equation that as \( \varepsilon \) goes to zero,

\[
det\left(\hat{s}_i I - \sqrt{\varepsilon}A + W \hat{P}_\varepsilon\right) \to 0, \quad i = 1, 2, ..., n
\] (46)
Substituting $P_\varepsilon = \sqrt{\varepsilon} \hat{P}_\varepsilon$ in the Riccati equation (44) yields
\[-\sqrt{\varepsilon} \hat{P}_\varepsilon A - \sqrt{\varepsilon} A^T \hat{P}_\varepsilon - Q + \hat{P}_\varepsilon W \hat{P}_\varepsilon = 0\]
Since the solution of the Riccati equation as well as the matrices $W$ and $Q$ are all positive definite, according to Lemma 1, as $\varepsilon$ approaches zero, $P_\varepsilon$ converges to a unique positive definite matrix denoted by $\hat{P}$, which can be obtained by solving the equation $Q = PW \hat{P}$, as discussed in Lemma 1. In other words, as $\varepsilon$ goes to zero, the solution of the Riccati equation $P_\varepsilon$ for the system $S$ and the performance index (32) can be estimated by $\sqrt{\varepsilon} \hat{P}$. Accordingly, since $\hat{P}_\varepsilon$ converges to $\hat{P}$ as $\varepsilon$ approaches zero, the equation (46) yields
\[
det \left( \hat{s}_i I + W \hat{P} \right) = 0, \quad i = 1, 2, ..., n \tag{47}\]
Now, consider the modified system $S^\varepsilon$ under the feedback law (33), and let the corresponding closed-loop modes be denoted by $\sigma_{i1}, \sigma_{i2}, ..., \sigma_{n_j}$. It is clear that these modes satisfy the following equation:
\[
det \left( \sigma_{ij} I - \tilde{A} + \frac{1}{\varepsilon} WP_\varepsilon \right) = 0, \quad i = 1, 2, ..., n \tag{48}\]
The above discussion shows that as $\varepsilon$ goes to zero, $P_\varepsilon$ converges to $\sqrt{\varepsilon} \hat{P}$. Therefore, it can be concluded from the equation (48) that (as $\varepsilon$ approaches zero):
\[
det \left( \sqrt{\varepsilon} \sigma_{ij} I - \sqrt{\varepsilon} \hat{A} + WP_\varepsilon \right) \rightarrow 0, \quad i = 1, 2, ..., n \tag{49}\]
By comparing equations (47) and (49), it can be concluded that as $\varepsilon$ goes to zero, the elements of the set $\{\sqrt{\varepsilon} \sigma_{i1}, ..., \sqrt{\varepsilon} \sigma_{nj}\}$ converge to the elements of the set $\{\hat{s}_1, ..., \hat{s}_n\}$. According to Theorem 6, $\hat{s}_1, ..., \hat{s}_n$ are all negative numbers. Thus, $\sqrt{\varepsilon} \sigma_{i1}, ..., \sqrt{\varepsilon} \sigma_{nj}$ will go to $n$ negative real numbers. As a result, as $\varepsilon$ approaches zero, all of these modes will move towards the left-half s-plane, and eventually all of them will be located in the open left-half s-plane. Thus, from continuity, one can conclude that there is a positive value $\varepsilon^*$ such that for every $\varepsilon < \varepsilon^*$, all complex numbers $\sigma_{i1}, ..., \sigma_{nj}$ will have negative real parts, and hence, the resultant closed-loop system will be stable.

Remark 8: As $\varepsilon$ approaches zero, $\sqrt{\varepsilon} \sigma_{i1}, \sqrt{\varepsilon} \sigma_{i2}, ..., \sqrt{\varepsilon} \sigma_{nj}$ converge to $n$ finite negative real numbers. Thus, $\sigma_{i1}, \sigma_{i2}, ..., \sigma_{nj}$ all go to $-\infty$.

Remark 9: Since the elements of both sets $\{\sqrt{\varepsilon} \sigma_{i1}, ..., \sqrt{\varepsilon} \sigma_{nj}\}$ and $\{\sqrt{\varepsilon} s_{i1}, ..., \sqrt{\varepsilon} s_{nj}\}$ approach the elements of the set $\{\hat{s}_1, ..., \hat{s}_n\}$ as $\varepsilon$ goes to zero, it can be deduced that the modes of the modified system $S^\varepsilon$ under the feedback law (33) become closer to the modes of the original system $S$ under the feedback law (33), as $\varepsilon$ approaches zero.

Consider again the system represented by the state-space matrices (39). Solving the centralized optimal LQR problem for $R = Q = I$ and multiplying the eigenvalues of the resultant closed-loop modified system $S^\varepsilon$ (under the feedback law (33)) by $\sqrt{\varepsilon}$ as described in Theorem 7, i.e. $\{\sqrt{\varepsilon} \sigma_{12}, \sqrt{\varepsilon} \sigma_{22}, \sqrt{\varepsilon} \sigma_{32}, \sqrt{\varepsilon} \sigma_{42}\}$, the following results are obtained for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$ and $10^{-5}$, respectively:
\[
\begin{align*}
\{&60.467, -10.607, -3.4948, -2.5690\}, &\{&60.352, -10.607, -2.0194, -2.5533\}, \\
&\{&60.340, -10.608, -2.5460, -1.8143\}, &\{&60.339, -10.608, -2.5454, -1.7923\}
\end{align*}\tag{50}\]
Comparing (50) and (41), it can be seen that as $\varepsilon$ goes to zero, the eigenvalues of the modified system $S^\varepsilon$ under the feedback law (33) multiplied by $\sqrt{\varepsilon}$ converge to the negative elements of the set given by (41), as expected.

Corollary 2: Suppose that $\det(W) \neq 0$. If $\varepsilon$ is sufficiently close to zero, the system $S$ under the proposed control law (33) is stable.

Proof: It can be concluded from Theorem 7 that there is an $\varepsilon^*$ such that for every positive real value $\varepsilon < \varepsilon^*$, all of the systems $S^1, S^2, ..., S^v$ are stable under the feedback law (33). Therefore, according to Theorem 2, the proposed decentralized feedback law stabilizes the system $S$ (for any $0 < \varepsilon < \varepsilon^*$).

Remark 10: To investigate robust stability of the proposed decentralized cheap control law, one can use the result of Theorem 3 to find the permissible range of parameter variations. As a particular case, assume that $\det(W') \neq 0$ and that $B_i = B_i$ for $i = 1, 2, ..., \nu$, i.e. there is no perturbation in the entries of the matrix $B$. It was shown that as $\varepsilon$ approaches zero, the modes of the modified system $S^i$ under the feedback law (33) $(\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in})$ converge to $\frac{1}{\sqrt{\varepsilon}}$ times the numbers $\hat{s}_1, \hat{s}_2, ..., \hat{s}_n$, which are obtained for the given $B, R$ and $Q$ using (34). In other words, dependency of the eigenvalues of the modified system $S^i$ under the feedback law (33) on the entries of the matrix $A$ is being reduced, as $\varepsilon$ goes to zero. Consider now the modified perturbed system $S^\varepsilon$. The only difference between $S^\varepsilon$ and $S^i$ is in the matrices $A^\varepsilon$ and $A^i$, or more specifically, in $A_{ii}$ and $A_{iji}$. Hence, as discussed before, the discrepancy between the modes of $S^\varepsilon$ and $S^i$ under the
feedback law (33) is reduced, as $\varepsilon$ approaches zero. This means that as $\varepsilon$ goes to zero, the eigenvalues of the perturbed system $S$ under the proposed local controllers become insensitive to the entries of the matrix $A$.

Remark 11: It is to be noted that the condition $\det(W) \neq 0$ is equivalent to $\det(BB^T) \neq 0$, or equivalently $\det(B_iB_i^T) \neq 0$ for any $i \in \nu$. Therefore, if the number of inputs of any subsystem is less than the number of its outputs, then the matrix $B_iB_i^T$ will be singular, and consequently the condition of Theorem 6 will be violated. Although this condition on the number of inputs of each subsystem can be very restrictive in general, in many practical problems it can be satisfied by adding certain actuators to some of the subsystems, if necessary.

VIII. NUMERICAL EXAMPLES

In this section, two examples will be presented. The first one is a numerical example which aims to illustrate some of the procedures developed in the paper. The second one applies the results obtained in this paper, to the formation flying problem in [2], and involves simulations.

Example 1: Consider a system $S_0$ consisting of two SISO subsystems and the following state-space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ -20 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The modified system $S_0$ is obtained by removing the interconnection going to the second subsystem (i.e. by setting the entry $-20$ of $A$ to zero). Suppose that $K_e$ is the optimal feedback gain for the system $S$ and the performance index (32), with $R = Q = I$. According to Theorem 7, since $\det(BR^{-1}B^T) \neq 0$, there exists a positive real $\varepsilon^*$ such that for every positive $\varepsilon < \varepsilon^*$, the modified system $S_0$ under the feedback law $u(t) = -K_e x(t)$ is stable. Computing $K_e$ for $\varepsilon = 1$, the eigenvalues of the modified system $S_0$ under the feedback law $u(t) = -K_e x(t)$ are obtained to be 0.2169 and $-7.1087$. According to Theorem 2, since one of these eigenvalues is positive, the overall closed-loop system is unstable. Therefore, $\varepsilon^*$ has to be less than one. It can be verified that for this example $\varepsilon^* \simeq 0.668$. Hence, for every $\varepsilon < 0.668$, the proposed local controllers can stabilize the system $S_0$.

Let $\varepsilon$ be equal to 0.001. Computing $K_e$ for this value of $\varepsilon$, it can be shown that the eigenvalues of the system $S_0$ and the modified system $S_0$ under the feedback law $u(t) = -K_e x(t)$ are stable. According to the

Example 2: Consider the perturbed system $S_{\varepsilon}$ under the proposed decentralized control law for $\varepsilon = 0.001$, and compare it to the robustness of the system $S_0$ under the centralized feedback law $u(t) = -K_e x(t)$.

1) Decentralized case: According to Theorem 3, the perturbed system $S_{\varepsilon}$ under the proposed decentralized controller is stable if the modified perturbed systems $S_0$ and $S_2$ under the feedback law $u(t) = -K_e x(t)$ are both stable. Therefore, any $s$ which satisfies one of the following equations:

$$\det(sI - \bar{A}^1 + \bar{B}^1 K) = 0, \quad \det(sI - \bar{A}^2 + \bar{B}^2 K) = 0$$

should have a negative real part. It is desired now to find some relations which exhibit the maximum allowable deviations from the nominal parameters of the system. Define:

$$\Delta A_{ij} := \bar{A}_{ij} - A_{ij}, \quad i, j \in \{1, 2\}, \quad i \geq j$$

$$\Delta B_i := \bar{B}_i - B_i, \quad i = 1, 2$$

Since all of the roots of the equations given in (51) should be in the left-half $s$-plane, it is easy to verify that the allowable perturbations are given by the following inequalities:

$$32.366 \Delta A_{11} - \Delta A_{11} > -33.366, \quad 32.013 \Delta B_1 - \Delta A_{11} > -95.915,$$

$$31.951 \Delta B_2 - \Delta A_{22} > -95.915, \quad -31.278 \Delta B_2 - \Delta A_{22} > -61.557, \quad \Delta A_{21} = \text{arbitrary}$$

(52)

2) Centralized case: Consider the perturbed system $S_{\varepsilon}$ under the feedback law $u(t) = -K_e x(t)$. The closed-loop system is stable, iff all of the roots of the equation $\det(sI - \bar{A} + \bar{B} K) = 0$ have negative real parts. Hence, the allowable ranges of perturbations in the centralized case satisfy the following inequalities:

$$-9.910 \Delta A_{22} - 18.88 \Delta A_{11} - \Delta A_{21} + 611.163 \Delta B_1 + 309.997 \Delta B_2 + 0.3 \Delta A_{11} \Delta A_{22} - 9.591 \Delta A_{11} \Delta B_2$$

$$\Delta A_{11} - \Delta A_{22} + 32.013 \Delta B_1 + 31.951 \Delta B_2 > -95.915$$

To compare robustness of the decentralized and centralized controllers, suppose that $\Delta B_1 = \Delta B_2 = 0$. According to the inequalities in (52), the admissible parameter variations in the decentralized case are as follows:

$$\Delta A_{11} < 33.366, \quad \Delta A_{22} < 61.557, \quad \Delta A_{21} < +\infty$$

(53)
The admissible parameter variations in the centralized case, on the other hand, are given by:
\[
9.910 \Delta A_{22} + 18.88 \Delta A_{11} + \Delta A_{21} - 0.3 \Delta A_{11} \Delta A_{22} < 630.045 \\
\Delta A_{11} + \Delta A_{22} < 95.915
\]
(54a) (54b)

From (53) and (54), it is clear that the centralized controller is less robust to the parameter variations compared to its decentralized counterpart, because:

- Stability in the decentralized case is independent of \( \Delta A_{21} \) but in the centralized case it is not.
- Regardless of \( \Delta A_{21} \), there is the term \( \Delta A_{11} \Delta A_{22} \) in the centralized case. This implies that when the two perturbations \( \Delta A_{11} \) and \( \Delta A_{22} \) have the same sign, (54a) can be easily violated, even if \( \Delta A_{21} \) is zero.

**Example 2:** Consider a leader-follower formation control system consisting of three vehicles. Assume that the state of each vehicle is available for its follower vehicle in order to generate the corresponding control signal. The objective is to design a controller such that all of the vehicles fly at the same desired speed, while the desired Euclidean distances between vehicles are achieved. The exact linearized model for the aforementioned problem (for certain specifications for each vehicle) is obtained in [2], and the tracking problem is converted to a regulation problem given below.

Consider an interconnected system consisting of three subsystems with the following state-space representation:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0_2 & 0_2 & 0_2 \\
I_2 & 0_2 & 0_2 \\
0_2 & 0_2 & I_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0_2 & 0_2 & 0_2 \\
I_2 & 0_2 & 0_2 \\
0_2 & 0_2 & I_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
(55)

where \( I_2 \) and \( 0_2 \) represent a 2 \times 2 identity matrix and a 2 \times 2 zero matrix, respectively, and

\[
x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix}, \quad x_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \\ x_{34} \end{bmatrix}
\]
(56)

and where \( u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} \), \( i = 1, 2, 3 \). Here, \( x_1 \) denotes the state of the leader, and \( x_2 \) and \( x_3 \) represent the state of vehicles 2 and 3 (i.e., the followers), respectively. More specifically:

1) \( x_{11} \) and \( x_{12} \) are the speed error of the leader (speed of the leader minus its desired speed) along \( x \) and \( y \) axes, respectively.
2) \( x_{i2} \), \( i = 2, 3 \), are the distance error (distance between vehicles \( i \) and \( i-1 \) minus their desired distance) along \( x \) and \( y \) axes, respectively.
3) \( x_{i3} \) and \( x_{i4} \), \( i = 2, 3 \), are the speed error (speed of vehicle \( i \) minus its desired speed) along \( x \) and \( y \) axes, respectively.
4) \( u_{i1} \) and \( u_{i2} \), \( i = 1, 2, 3 \), are the acceleration of vehicle \( i \) along \( x \) and \( y \) axes, respectively.

It is desired now to design a decentralized controller for the system given by (55), such that the closed-loop system is stable. Moreover, the objective is that the state variables of the closed-loop system decay as sharply as possible, with a reasonably small control effort. To attain these specifications, consider the performance index given by (3) in the paper, and assume that \( Q = R = I \). Two different design techniques will be used and the results will be compared here: the iterative numerical procedure given in [17], and the method proposed in this paper. Suppose that each initial state is uniformly distributed in the intervals \([0, 0.4]\), and that any two distinct initial state variables are statistically independent. It is to be noted that the units used for distance and velocity in the state vectors are ft and ft/s, respectively. Assume that any two different subsystems consider the same expected value for the initial state of the remaining subsystem, and that the model of each subsystem is exactly known by the other subsystems. It can be concluded from Procedure 1 and Remark 7 that if the real initial state variables are close to their expected value 300, the controller obtained by using the proposed method performs better.

Assume that the real initial state variables are all equal to 400, which correspond, in fact, to the worst case scenario (maximum discrepancy between the real initial state variables, i.e. 400, and the corresponding expected values, i.e. 300, which are used by the proposed controller). The iterative numerical procedure of [17] gives a static decentralized state feedback law which results in a performance index equal to 2,257,085. The performance index obtained by applying the method proposed in this paper, on the other hand, is equal to 2,090,939, while the best achievable performance index corresponding to the centralized LQR controller is equal to 2,068,513. This means that the relative errors of the performance indices obtained by using the methods given here and in [17], with respect to the optimal centralized performance index are 1.08\% and 9.12\%, respectively. This shows clearly that the controller proposed in this paper outperforms the one presented in [17], significantly.
Figures 1 depict the time responses of the system under the controller proposed in this paper (dotted curve), the controller proposed in [17] (dashed curve), and the optimal centralized controller (solid curve) for three state variables $x_{11}, x_{31}, x_{33}$. Moreover, the control signals $u_{11}, u_{21}, u_{31}$ obtained by using the three methods discussed above is depicted in Figure 2 in a similar way. It is to be noted that despite the relatively big differences between the real initial variables (400 ft for distance errors and 400 ft/sec for speed errors) and the corresponding expected values which are used to construct the proposed controller, the results obtained are reasonably close to the time response of the system under the LQR controller.

The results obtained show that the method introduced in present work is much better than the one in [17]. On the other hand, as stated in the introduction, the controller obtained by the method in [17] has a better performance compared to the ones proposed in [14], [15], and [16]. In addition, the control law given in [24] can potentially outperform the ones in [14], [15], and [16], but can never perform better than the one in [17]. This exhibits superiority of the proposed design technique over the existing ones.

**IX. CONCLUSIONS**

In this paper, an incrementally linear decentralized control law for the formation of vehicles with leader-follower structure is introduced. The fundamental idea in constructing this control law is that the local controller of each vehicle exploits \textit{a priori} information about the models and the expected values of all other vehicles. It is shown that the decentralized closed-loop system can behave the same as the optimal centralized closed-loop system (with respect to a quadratic performance index) if the \textit{a priori} knowledge of each subsystem is perfect. Since this knowledge can be inaccurate, the performance degradation of the resultant decentralized closed-loop system has been evaluated thoroughly, in presence of inexact information. The proposed decentralized control strategy is very easy to implement, and the corresponding stability verification steps are very easy to check as illustrated in the examples. Furthermore, it is shown that the decentralized control system is, in general, more robust than its centralized counterpart. Optimal decentralized cheap control problem is investigated for leader-follower formation structure, and a closed-form solution is provided for the case when the input structure meets a certain condition. This can be very useful for \textit{UAV} missions with fast tracking objectives. Simulation results demonstrate the effectiveness of the proposed method compared to the existing ones.
REFERENCES