Robust Stability of LTI Discrete-Time Systems using Sum-of-Squares Matrix Polynomials

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This note presents the main results of the paper "Robust stability of LTI discrete-time systems using sum-of-squares matrix polynomials", to appear in the 2006 American Control Conference, paper ThC09.4.

I. MAIN RESULTS

Consider the following discrete-time system:

\[ x(\kappa + 1) = A(\alpha)x(\kappa), \quad \kappa = 0, 1, 2, \ldots \]  \hspace{1cm} (1)

where \( \alpha \) denotes the vector of coefficients \( \left[ \alpha_1 \alpha_2 \cdots \alpha_n \right] \), \( x(\kappa) \in \mathbb{R}^n \) is the state, and \( A(\alpha) \in \mathbb{R}^{n\times n} \) represents the perturbation of the system matrix. Let \( A(\alpha) \) be written as

\[ A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n \]  \hspace{1cm} (2)

where \( A_i \in \mathbb{R}^{n\times n}, i \in \{1, 2, \ldots, n\} \) are known matrices, and:

\[ 0 \leq \alpha_i \leq 1, \quad i \in \{1, 2, \ldots, n\}, \quad \sum_{i=1}^{n} \alpha_i = 1 \]  \hspace{1cm} (3)

It is desired to find stability conditions for the system (1), for any vector \( \alpha \) satisfying (3).

The system (1) is robust stable in the domain given by (3) if and only if there exists a symmetric matrix \( P(\alpha) \) such that the matrix

\[ \Phi(\alpha) := \begin{bmatrix} P(\alpha) & A^T(\alpha)P(\alpha) \\ P(\alpha)A(\alpha) & P(\alpha) \end{bmatrix} \]  \hspace{1cm} (4)

is positive definite for any \( \alpha \) belonging to the domain (3).

**Definition 1:** A polynomial of the form \( E(\omega) = \sum_{i_1, \ldots, i_l} E_{i_1 \ldots i_l} \omega_1^{i_1} \omega_2^{i_2} \cdots \omega_l^{i_l} \), where \( \omega = [\omega_1 \omega_2 \cdots \omega_l] \) and the coefficients \( E_{i_1 \ldots i_l} \) are matrices of proper dimensions (instead of scalars), is defined to be a matrix polynomial. It is to be noted that the variables \( \omega_1, \ldots, \omega_l \) are scalar. In the special case, when the coefficients \( E_{i_1 \ldots i_l} \) are scalars, the corresponding polynomial is called a scalar polynomial.

**Notation 1:** The bold symbol is used throughout this paper to represent a vector of scalar variables corresponding to a matrix polynomial.

**Definition 2:** Each product term of a scalar polynomial (or a matrix polynomial) \( c(\omega) \), where \( \omega = [\omega_1 \omega_2 \cdots \omega_l] \), is defined to be a monomial of \( c(\omega) \). In general, a monomial has the form \( \omega_1^{i_1} \omega_2^{i_2} \cdots \omega_l^{i_l} \), where \( i_1, i_2, \ldots, i_l \) are nonnegative integers. For instance, the monomials of \( 3\omega_1 - \omega_1\omega_2 + 5 \) are \( \omega_1, \omega_1\omega_2 \) and 1.

**Notation 2:** Divisibility of a scalar or matrix polynomial \( c_1(\omega) \) by a scalar polynomial \( c_2(\omega) \) is denoted by \( c_1(\omega)|c_2(\omega) \).

**Definition 3:** A matrix polynomial \( C(\omega) \) with the scalar variables \( \omega_1, \ldots, \omega_l \) is defined to be a sum-of-squares (SOS) if there exists a matrix polynomial \( E(\omega) \) such that

\[ C(\omega) = E^T(\omega)E(\omega) \]  \hspace{1cm} (5)

**Theorem 1:** (Theorem 2 in [6]) Consider a symmetric matrix polynomial \( H(\omega) \) and the scalar polynomials \( h_1(\omega), h_2(\omega), \ldots, h_k(\omega) \), where \( \omega \in \mathbb{R}^l \), and assume that there exist a scalar \( r \) and scalar SOS polynomials \( g_0(\omega), g_1(\omega), \ldots, g_k(\omega) \) such that

\[ r^2 - \omega^T = g_0(\omega) + \sum_{i=1}^{k} g_i(\omega)h_i(\omega) \]  \hspace{1cm} (6)

The matrix polynomial \( H(\omega) \) is positive definite for any value of \( \omega \) belonging to \( \{\omega \in \mathbb{R}^l | h_1(\omega) \geq 0, h_2(\omega) \geq 0, \ldots, h_k(\omega) \geq 0\} \), if and only if there exist SOS matrix polynomials \( Y_0(\omega), Y_1(\omega), \ldots, Y_k(\omega) \) and a positive scalar \( \varepsilon > 0 \) such that:

\[ H(\omega) = Y_0(\omega) + \sum_{i=1}^{k} h_i(\omega)Y_i(\omega) + \varepsilon I \]  \hspace{1cm} (7)

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symmetric matrix polynomials $Q_1(\omega)$ and $Q_2(\omega)$, where $\omega = [\omega_1 \omega_2 \cdots \omega_n]$, such that $Q_2(\omega)$ is SOS, and
\[ \Phi(\omega^2) = (1 - \omega \omega^T) Q_1(\omega) + Q_2(\omega) + \varepsilon I_{2\nu} \]  
(9)
for all $\omega_i \in \mathbb{R}$, $i = 1, 2, \ldots, \nu$ ($I_{2\nu}$ denotes the $2\nu \times 2\nu$ identity matrix).

**Proof of necessity:** Assume that the system $S$ is robust stable in the domain (3). Since the Lyapunov function $P(\alpha)$ can be assumed to be a matrix polynomial [7], the function $\Phi(\alpha)$ can be considered as a symmetric polynomial. Define the $(i, j)$ entry of $\Phi(\alpha)$, which is a scalar polynomial, as $\phi_{ij}(\alpha)$ for any $i, j \in \{1, 2, \ldots, 2\nu\}$. It is obvious that
\[ 1 - (\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n) \phi_{ij}(\alpha) - \phi_{ij}(\bar{\alpha}) \]  
(10)
for any $i, j \in \{1, 2, \ldots, 2\nu\}$ (the notation $\bar{\alpha}$ is introduced in (5)). Therefore,
\[ 1 - (\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n) \left| \Phi(\alpha) - \Phi(\bar{\alpha}) \right| \]  
(11)
Consider now the arbitrary scalars $\omega_1, \omega_2, \ldots, \omega_n$. Substituting $\alpha_i = \omega_i^2$, $i = 1, 2, \ldots, \nu$, into (11) yields
\[ 1 - (\omega_1^2 + \cdots + \omega_{n-1}^2 + \omega_n^2) \left| \Phi(\omega^2) - \Phi(\overline{\omega^2}) \right| \]  
(12)
Note that
\[ \overline{\omega^2} = [\omega_1^2 \cdots \omega_{n-1}^2 1 - (\omega_1^2 + \cdots + \omega_{n-1}^2)] \]  
(13)
as defined in (5). It results from (12) that there exists a matrix polynomial $G_1(\omega)$ such that
\[ \Phi(\omega^2) = (1 - \omega \omega^T) G_1(\omega) + \Phi(\overline{\omega^2}) \]  
(14)
It is to be noted that since $\Phi(\alpha)$ is symmetric, $G_1(\omega)$ is symmetric as well. On the other hand, since $\Phi(\alpha)$ is positive definite for any $\alpha$ belonging to the domain (3), $\Phi(\overline{\omega^2})$ is positive definite for any $\omega_1, \omega_2, \ldots, \omega_{n-1}$ satisfying the inequality $0 \leq 1 - (\omega_1^2 + \cdots + \omega_{n-1}^2)$ (because $\omega_1^2 + \cdots + \omega_{n-1}^2 + [1 - (\omega_1^2 + \cdots + \omega_{n-1}^2)] = 1$). Thus, by considering $h_1(\overline{\omega}) = 1 - \omega_1^2 - \cdots - \omega_{n-1}^2$, $g_1(\overline{\omega}) = 0$, $g_1(\overline{\omega}) = 1$, it can be concluded from Lemma 1 that there exist a scalar $\varepsilon > 0$ and two SOS matrix polynomials $G_2(\overline{\omega})$ and $G_3(\overline{\omega})$ such that
\[ \Phi(\overline{\omega^2}) = \left[ 1 - (\omega_1^2 + \cdots + \omega_{n-1}^2) \right] G_2(\overline{\omega}) + G_3(\overline{\omega}) + \varepsilon I_{2\nu} \]  
(15)
Substituting (15) into (14) yields
\[ \Phi(\omega^2) = (1 - \omega \omega^T) G_1(\omega) + \varepsilon I_{2\nu} \]  
(16)
\[ + (1 - (\omega_1^2 + \cdots + \omega_{n-1}^2)) G_2(\overline{\omega}) + G_3(\overline{\omega}) \]  
\[ = (1 - \omega \omega^T) \left[ G_1(\omega) + G_2(\overline{\omega}) \right] \]  
\[ + \left[ \omega_2^2 G_2(\overline{\omega}) + G_3(\overline{\omega}) \right] + \varepsilon I_{2\nu} \]
Define now
\[ Q_1(\omega) := G_1(\omega) + G_2(\overline{\omega}), \quad Q_2(\omega) := \omega_2^2 G_2(\overline{\omega}) + G_3(\overline{\omega}) \]  
(17)
It is clear that $Q_1(\omega)$ and $Q_2(\omega)$ satisfy (9) (according to (16)). However, it is required to show that $Q_2(\omega)$ defined in (17) is SOS. Since $G_2(\overline{\omega})$ and $G_3(\overline{\omega})$ are SOS, as discussed earlier, there exist two constant positive semidefinite matrices $\Lambda_1$ and $\Lambda_2$ such that
\[ G_2(\overline{\omega}) = \Omega(\overline{\omega}) \Lambda_1 \Omega(\overline{\omega})^T, \quad G_3(\overline{\omega}) = \Omega(\overline{\omega}) \Lambda_2 \Omega(\overline{\omega})^T \]  
(18)
where $\Omega(\overline{\omega})$ is a block vector whose block entries are all monomials of $G_2(\overline{\omega})$ and $G_3(\overline{\omega})$ times an identity matrix with the proper dimension. Therefore,
\[ Q_2(\omega) = \left[ \begin{array}{ccc} \Omega(\overline{\omega}) & \omega_2 \Omega(\overline{\omega}) & 0 \\ \omega_2 \Omega(\overline{\omega})^T & 0 & \Lambda_1 \\ \omega_2 \Omega(\overline{\omega}) & \omega_2 \Omega(\overline{\omega})^T & \Omega(\overline{\omega})^T \end{array} \right] \]  
(19)
Since $\Lambda_1$ and $\Lambda_2$ are positive semidefinite, it can be concluded from the above relation (by writing the semidefinite matrix in the above expression as the square of a matrix) that $Q_2(\omega)$ is SOS. It is to be noted that since $G_2(\overline{\omega})$ and $G_3(\overline{\omega})$ are SOS, they are symmetric as well. On the other hand, it is shown that $G_1(\omega)$ is also symmetric. As a result, $Q_1(\omega)$ and $Q_2(\omega)$ are both symmetric.

**Proof of sufficiency:** Suppose that there exist a positive scalar $\varepsilon$ and two symmetric matrix polynomials $Q_1(\omega)$ and $Q_2(\omega)$ such that $Q_2(\omega)$ is SOS and the equality (9) holds for any real values $\omega_1, \omega_2, \ldots, \omega_n$. Consider now the scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying the conditions given by (3). It is clear that there exist scalars $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_i^2 = \alpha_i$, $i = 1, 2, \ldots, n$, which yields $\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2 = 1$. Thus, one can conclude from (9) that:
\[ \Phi(\alpha) = \Phi(\omega^2) = Q_2(\omega) + \varepsilon I_{2\nu} \]  
(16)
The proof follows from the fact that $\varepsilon > 0$ and by noting that $Q_2(\omega)$ is assumed to be SOS and hence positive semidefinite.

**Corollary 1:** The system $S$ is robust stable in the domain (3) if and only if there exist three symmetric matrix polynomials $\hat{P}(\omega)$, $Q_1(\omega)$ and $Q_2(\omega)$, where $\omega = [\omega_1 \omega_2 \cdots \omega_n]$, such that $Q_2(\omega)$ is SOS, and
\[ \left[ \begin{array}{c} \hat{P}(\omega^2) \\ A^T(\omega^2) \hat{P}(\omega^2) \\ \hat{P}(\omega^2) A(\omega^2) \hat{P}(\omega^2) \end{array} \right] = \tilde{Q}_2(\omega) \]  
(21)
for all $\omega_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

**Proof of necessity:** Consider the matrix polynomials $P(\alpha)$ (or $\Phi(\omega)$), $Q_1(\omega)$ and $Q_2(\omega)$, and the positive scalar $\varepsilon$ in Theorem 1. Define now the following matrix polynomials:
\[ \hat{P}(\alpha) := \frac{P(\alpha)}{\varepsilon}, \quad \hat{Q}_1(\omega) := \frac{Q_1(\omega)}{\varepsilon}, \quad \hat{Q}_2(\omega) := \frac{Q_2(\omega)}{\varepsilon} \]  
(22)
It is straightforward to verify that $\hat{Q}_2(\omega)$ is SOS, and also, the matrix polynomials $\hat{P}(\omega)$, $\hat{Q}_1(\omega)$ and $\hat{Q}_2(\omega)$ satisfy the relation (21) (by using the result of Theorem 1). The proof of sufficiency is omitted due to its similarity to that of Theorem 1.

It can be concluded from Corollary 1 that by assuming some bounds on the degrees of the polynomials
\( \hat{P}(\omega), \hat{Q}_1(\omega) \) and \( \hat{Q}_2(\omega) \), one can denote the unknown coefficients of these polynomials with some variables, expand both sides of the equation (21), and equate the corresponding coefficients in order to obtain a set of equality constraints in terms of the relevant variables (the coefficients). These constraints along with the condition \( \hat{Q}_2(\omega) \geq 0 \) establish a SDP problem [8]. The resultant problem can then be solved by using a proper software [9].

REFERENCES