Self-Deployment Algorithms for Coverage Problem in a Network of Mobile Sensors with Unidentical Sensing Range

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Abstract

In this paper, efficient sensor deployment algorithms are proposed to improve the coverage area in the target field. The proposed algorithms calculate the position of the sensors iteratively based on the existing coverage holes in the target field. The multiplicatively weighted Voronoi diagram (MW-Voronoi diagram) is used to discover the coverage holes corresponding to different sensors with different sensing ranges. Under the proposed procedures, the sensors move in such a way that the coverage holes in the target field are reduced. Simulation results are provided to demonstrate the effectiveness of the deployment schemes proposed in this paper.

I. INTRODUCTION

Wireless sensor networks have attracted considerable attention in the literature in recent years, due to their widespread applications in civilian and military environments [1], [2], [3], [4], [5]. Examples of such applications include robot-assisted sensor networks for data collection [6], security and surveillance [7], and using mobile sensors in toxic areas [8], to name only a few. One of the main area of interest in this type of system is concerned with the development of efficient sensor deployment strategies to improve both coverage and resource management in the target field [9], [10]. There are a number of practical constraints which need to be taken into account in designing control algorithms for sensor networks. For instance, in many real-world applications no a priori knowledge is available about the initial position of the sensors. Furthermore, it is often desirable to have some form of decentralized strategy due to the distributed nature of the system. In other words, each sensor is required to make a decision based on its limited communication and sensing capability and limited knowledge obtained from other sensors.

A Voronoi-based approach for field coverage is presented in [11], where no global location assurance condition is required for the sensors. The problem of field coverage for a group of sensors following any given trajectory is investigated in [12]. A multi-objective deployment and power assignment algorithm is proposed in [13], where the optimization problem is decomposed into several scalar single-objective problems which are to be solved simultaneously. In [14], an algorithm is presented to add a relatively small number of mobile sensors to a set of static sensors in order to improve the field coverage. The algorithm employs a strategy which aims to optimize the contribution of the mobile sensors to the field coverage.

In [15], [16] robotic sensor networks performing coverage optimization tasks with area constraints is studied. In [17], distributed control policies are presented to allow a team of agents to achieve a convex equi-partition configuration. In [18] sensor locations which optimize the maximum error variance and extended prediction variance are determined. Distributed control laws for disk-covering and sphere-packing problems using nonsmooth gradient flows proposed in [19]. In [20] an algorithm is proposed to monitor an environmental boundary with mobile agents. The boundary is optimally approximated with a polygon. Decentralized control laws for the optimal positioning of sensor networks for target tracking are presented in [21].

In [8], [22], [23], it is assumed that the sensing radii of the sensors are equal, and the Voronoi diagram is used to partition the target field to find the coverage holes corresponding to each sensor. In [23], three algorithm, namely VEC, VOR, and Minmax are proposed to determine the final destination of the sensors. In [22], [23], two different approaches (basic protocols and virtual movement protocols) are introduced to deploy the sensors in appropriate locations to achieve a satisfactory coverage. In the above papers (and most of the existing results), the sensing capabilities of all sensors are assumed to be identical [8], [22], [23], [11].

In the present work, the problem of improving the coverage area in a network of mobile sensors with different sensing capabilities is studied. The multiplicity weighted Voronoi (MW-Voronoi) diagram is utilized to find the coverage holes, where the weight assigned to each sensor is proportional to its sensing radius [24], [25], [26]. Eight algorithms are proposed: the weighted vector based (WVB), farthest point boundary (FPB), Maxmin-vertex, Minmax-vertex, Minmax-curve, Maxmin-curve, Minmax-point and Curtex algorithms. The main idea behind these algorithms is to move each sensor iteratively in a direction that its coverage is improved. Once a new location for a sensor is computed, the corresponding local coverage area of the sensor (in the previously constructed MW-Voronoi diagram) is compared to the preceding local coverage area. If the new local coverage area is larger than the preceding one, the sensor moves to the new location; otherwise, it remains

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in the current position. If the local coverage area by each sensor in an iteration does not exceed a certain threshold, the algorithm is terminated (to ensure a finite number of iterations).

In WVB algorithm, sensors are pushed away from a densely covered area. To this end, the weighted average distance is defined and the distance between any pair of sensors is compared with a certain value, which is a function of the weights of the two sensors, and the weighted average distance. If the distance between the two sensors is less than this value, then they push each other by a virtual force. Finally, each sensor is moved by the vector summation of its corresponding virtual forces. In the FPB algorithm, each sensor moves to the farthest point in its MW-Voronoi region from where the existing coverage hole in its region can be covered. The Maxmin-vertex strategy determines the point inside the MW-Voronoi region which its distance from the closest Voronoi vertex is maximum. The Minmax-vertex strategy finds the point in a sensor’s MW-Voronoi region such that whose distance from the farthest Voronoi vertex becomes minimum. In Maxmin-curve strategy the sensor location is determined such that its distance from the nearest Voronoi curve be maximized. In Minmax-curve strategy find outs the next sensor’s location inside the MW-Voronoi region whose distance to the farthest Voronoi curve is minimized. The Minmax-point Strategy chooses a point inside the MW-Voronoi region as next sensor’s location whose distance to the farthest point of the region is minimized. In Curtex Strategy, two strategies Minmax-vertex and Maxmin-curve strategies are combined. In this algorithm sensors choose two destination location inside MW-Voronoi regions, and then they choose the one that has better coverage to move to that point. In the mentioned algorithms sensors move to their assigned locations at each iteration, until the termination condition is satisfied.

The rest of the paper is organized as follows. In Section II, some preliminaries and important notions and definitions are provided. Section III presents the main results of the paper, where new deployment algorithms are proposed. In Section IV, simulation results are given to demonstrate the effectiveness of the proposed approaches. Finally, concluding remarks are drawn in Section V.

II. BACKGROUND

A. Voronoi Diagram

Let $S$ be a set of $n$ distinct nodes in the 2-D plane, denoted by $S = \{S_1, S_2, \ldots, S_n\}$. The Voronoi diagram is a partitioning of the plane into a group of $n$ convex polygons, called Voronoi polygons, such that each polygon contains only one node and every point inside that polygon is closer to its generating node rather than to any other node in the plane [27], [28], [29], [30]. To construct the Voronoi diagram, first the perpendicular bisectors of the neighboring sensors are found for each sensor. The smallest polygon (created by the above perpendicular bisectors) containing each node is, in fact, the Voronoi polygon of that node. The Voronoi diagram is a basic tool for the analysis and design of sensor deployment techniques.

Represent each sensor in the field as a node and sketch the corresponding Voronoi polygons for all sensors as described above, to cover the entire target field. Recall that by definition any point inside a polygon is closer to its generating sensor than to any other sensor in the field. Thus, assuming that all sensors have the same sensing capability (in terms of coverage radius), if a sensor cannot detect a certain point inside its corresponding polygon, that point cannot be detected by any other sensor in the field either. This means that in order to identify the coverage holes, it suffices that each sensor checks its own Voronoi polygon to find the points it cannot cover. However, as noted earlier, this fundamental statement is only true for the case when all sensors have the same sensing range. It can be easily shown that when the sensing radii of different sensors are not the same, then a point which is not detected by the sensor corresponding to the polygon containing that point, may be detectable by the sensor corresponding to a neighboring polygon. Hence, when the sensors are not identical in terms of sensing radius, the conventional Voronoi diagram is not as useful for effective sensor deployment in the network. The MW-Voronoi diagram described in the next section is used to remedy this shortcoming.

B. MW-Voronoi Diagram

Let $S$ be a set of $n$ distinct weighted nodes in the plane denoted by $S = \{(S_1, w_1), (S_2, w_2), \ldots, (S_n, w_n)\}$, where $w_i > 0$ is the weighting factor associated with $S_i$, for any $i \in \bar{n} := \{1, 2, \ldots, n\}$. It is desired now to partition the plane into $n$ regions such that:

i) Each region contains only one node, and

ii) the nearest node, in the sense of weighted distance, to any point inside a region is the node assigned to that region.

The diagram obtained by the partitioning described above is called the multiplicatively weighted Voronoi diagram (MW-Voronoi diagram) [25]. Analogous to conventional Voronoi diagram, the mathematical characterization of each region obtained by the partitioning described above is as follows:

\[
\Pi_i = \{Q \in \mathbb{R}^2 | \ w_j d(Q, S_i) \leq w_i d(Q, S_j), \forall j \in \bar{n} - \{i\}\} \tag{1}
\]

for any $i \in \bar{n}$, where $d(Q, S_i)$ is the Euclidean distance between $Q$ and $S_i$.

According to (1), any point $Q$ in the $i$-th MW-Voronoi region $\Pi_i$, has the following property:

\[
\frac{d(Q, S_i)}{d(Q, S_j)} \leq \frac{w_i}{w_j}, \quad \forall i \in \bar{n}, \forall j \in \bar{n} - \{i\} \tag{2}
\]
Definition 1. Similar to conventional Voronoi diagram, the nodes \( S_i \) and \( S_j \) \((i, j \in n)\) in an MW-Voronoi diagram are called neighbors if \( \bar{\Pi}_i \cap \bar{\Pi}_j \neq \emptyset \). The set of all neighbors of \( S_i \), \( i \in n \), is denoted by \( N_i \) and is formulated below.

\[
N_i = \{ S_j \in S \mid \bar{\Pi}_i \cap \bar{\Pi}_j \neq \emptyset, \forall j \in n \}
\] (3)

Definition 2. Consider the sensor \( S_i \) with the sensing radius \( r_i \), and the corresponding MW-Voronoi region \( \bar{\Pi}_i \), \( i \in n \), and let \( Q \) be an arbitrary point inside \( \bar{\Pi}_i \). The intersection of the region \( \bar{\Pi}_i \) and a circle of radius \( r_i \) centered at \( Q \) is referred to as the coverage area with respect to (w.r.t.) \( Q \). The coverage area w.r.t. the location of the sensor \( S_i \) is called the local coverage area of that sensor.

Definition 3. The Apollonian circle of the segment \( AB \), denoted by \( \Omega_{AB,k} \), is the locus of all points \( E \) such that \( AE = BE = k \) [31].

To construct the \( i \)-th MW-Voronoi region, first the Apollonian circles of the neighboring partitions are found for the \( i \)-th sensor. In other words, the Apollonian circles \( \Omega_{S_iS_j} \) are found for all \( S_j \in N_i \). The smallest region (created by the above circles) containing the \( i \)-th node is, in fact, the \( i \)-th MW-Voronoi region (e.g., see Fig. 1). An example of a MW-Voronoi diagram with 15 sensors is sketched in Fig. 2.

![Fig. 1. The MW-Voronoi region for a sensor (S1) with four neighbors (S2 – S5).](image)

The MW-Voronoi diagram is the main tool for sensor deployment in this paper. Each sensor is characterized by a sensing area which is a circle whose size can be different for distinct sensors. Consider each sensor in the field as a node with a weight equal to its sensing radius, and sketch the MW-Voronoi region for each sensor; the resultant diagram covers the entire target field. From the characterization of the MW-Voronoi regions provided in (1), it is straightforward to show that if a sensor cannot detect a phenomenon in its corresponding region, no other sensor can detect it either. This means that in order to find the "so-called" coverage holes (i.e., the undetectable points in the target field), it would suffice to compare the MW-Voronoi region of each node with its local coverage area.

Definition 4. Consider a circle of radius \( r \) centered at \( O \), denoted by \( \Omega(O,r) \) hereafter, and a point \( V \) in the plane. The intersections of \( \Omega \) and the extension of \( VO \) from \( O \) is denoted by \( T_{1\Omega(O,r)}^V \). The other intersection point of \( \Omega(O,r) \) and \( VO \) (or its extension) is denoted by \( T_{2\Omega(O,r)}^V \).

Definition 5. As mentioned before, the boundary curves of an MW-Voronoi region are the segments of some Apollonian circles. The set of all such Apollonian circles for the \( i \)-th MW-Voronoi region is denoted by \( \Omega_i \). The sets \( \Omega_i \) and \( \tilde{\Omega}_i \) are defined as follows:

\[
\Omega_i = \{ \Omega \in \Omega_i \mid S_i \in \Omega \}
\]
\[
\tilde{\Omega}_i = \{ \Omega \in \Omega_i \mid S_i \notin \Omega \}
\]

Assumption 1. In this paper, it is assumed that there is no obstacle in the field. Therefore, the sensors can move to any desired location using existing techniques, e.g. [32], [33], [34].

Assumption 2. The sensors are supposed to be capable of localizing themselves with sufficient accuracy in the target field (using, for instance, the methods proposed in [35], [2]).
Assumption 3. The communication range of the sensors is bounded (and not necessarily the same for all sensors). This is a limiting factor for the sensors potentially preventing them from reaching their neighbors, which in turn results in a wrong Voronoi region around some of the sensors. Consequently, such a limitation can negatively affect the detection of coverage holes.

In this section, three different protocols are designed for a distributed sensor network. The proposed algorithms are iterative, where in each iteration every sensor first broadcasts its sensing radius and location to other sensors, and then constructs its MW-Voronoi region based on the received information. It checks the region subsequently to detect the possible coverage holes. If any coverage hole exists, the sensor calculates its target location (but does not move there) in such a way that the coverage hole is eliminated, or at least its size is reduced. Once the new target location is calculated, the coverage area w.r.t. the new target location (in the previously constructed MW-Voronoi region) is obtained. If this coverage area is greater than the current one, the sensor moves to the new location; otherwise it remains in its current position. In order to terminate the algorithm in finite time, one can define a coverage improvement threshold $\epsilon$ such that if the increase in the local coverage area by each sensor is not sufficiently large (as specified by $\epsilon$), there is no need to continue the iterations.

A. Weighted Vector Based Algorithm (WVB)

This algorithm tends to move the sensors out of the densely covered areas. Denote the distance between the sensors $S_i$ and $S_j$ with $d_{ij}$. Define a new (virtual) sensor network with $\bar{n} := \lceil \sum_{i=1}^{n} w_i^2 \rceil$ sensors with unit sensing radius, evenly distributed in the target area. Let $\bar{d}$ be the distance between a sensor and its nearest neighboring sensor in this new network (this distance can be calculated off-line). In this algorithm, if the distance between the two sensors $S_i$ and $S_j$ in the original network, $(i,j \in n)$, is less than $\frac{w_i + w_j}{2} \bar{d}$ and none of them covers its MW-Voronoi region completely, then a virtual force between the two sensors will push them as if it wants to move the sensor $S_i$ by $\frac{w_i}{w_i + w_j} D_{ij}$ and the sensor $S_j$ by $\frac{w_j}{w_i + w_j} D_{ij}$, where $D_{ij} = \frac{w_i + w_j}{2} \bar{d} - d_{ij}$. If, however, one of the two sensors, say $S_i$ covers its region completely, then the other sensor $S_j$ should only be pushed by the above-mentioned virtual force. In fact, for every pair of sensors, if there is a coverage hole in any of the corresponding two regions, then the virtual force tends to push the sensors away from each other by $\frac{w_i + w_j}{2} \bar{d}$.

On the other hand, virtual forces are also applied in a similar manner from the boundaries to those sensors which are close to them (compared to the weighted average distance). If the distance $d_{bi}$ between $S_i$ and a boundary is less than $\frac{w_i}{2} \bar{d}$, then a virtual force tends to push the sensor away from the boundary by $\frac{w_i}{2} \bar{d} - d_{bi}$. Eventually, each sensor is moved by the vector summation of all virtual forces from the boundaries and from other sensors.

B. Farthest Point Boundary Algorithm (FPB)

In this algorithm, each sensor moves to the farthest point in its MW-Voronoi region from where any existing coverage hole in its region can be covered. This point is denoted by $X_{i,\text{far}}$ for the $i$-th region. In fact, once a sensor detects a coverage hole, it calculates the farthest point (using the information about the MW-Voronoi region coordinates, as well as the coverage hole, as it will be shown later) and moves toward it and continues moving until $X_{i,\text{far}}$ is covered. The following definition is used to calculate the farthest point in each MW-Voronoi region.

**Definition 6.** The corner points of the $i$-th MW-Voronoi region (i.e., the intersection of its boundary curves) are denoted by $V_{i1}, V_{i2}, \ldots, V_{il}$, where $il \geq 0$. These points are called the MW-Voronoi vertices for the $i$-th MW-Voronoi region. It is to be noted that the farthest point in each MW-Voronoi region lies on the boundary of the region.
Lemma 1. Let $E$ and $F$ be two points on the circle $\Omega$, and $V$ be a point in the plane such that $T^V_{\Omega}$ is closer to $E$ than to $F$ (see Fig. 3). Then, $VE > VF$.

Proof: From the law of cosines in triangles $\Delta VOE$ and $\Delta VOF$, it yields that:

$$VE^2 = VO^2 + OE^2 - 2VO \times OE \times \cos \angle VOE$$
$$VF^2 = VO^2 + OF^2 - 2VO \times OF \times \cos \angle VOF$$

Since $0 \leq \angle VOF < \angle VOE \leq 180$, hence $\cos \angle VOE < \cos \angle VOF$. According to (4), (5) and from the relations $OE = OF$ and $\cos \angle VOE < \cos \angle VOF$ it can be concluded that $VE > VF$. This completes the proof. ■

Corollary 1. Given a positive constant $k \neq 1$, let $E$ and $F$ be two points on $\Omega_{AB,k}$ such that $T^A_{\Omega_{AB,k}}$ is closer to $E$ than to $F$ (see Fig. 4). Then, $AE > AF$ and $BE > BF$.

Proof: The proof follows immediately from Lemma 1, on noting that $AE = BE = k$.

Remark 1. It is implied from Lemma 1 that for any positive constant $k \neq 1$, $T^A_{\Omega_{AB,k}}$ is the farthest point to $A$ and $B$, and $\bar{T}^A_{\Omega_{AB,k}}$ is the nearest point to $A$ and $B$, between all points on $\Omega_{AB,k}$.

Lemma 2. Let $D$ be a point and $AB$ a segment in the plane. Between all points on $AB$, the farthest point from $D$ is either $A$ or $B$.

Proof: The proof is straightforward and is omitted here.

Theorem 1. Let $A_i$ be the set of all MW-Voronoi vertices for the $i$-th region, $i \in n$, and define the set $B_i$ as fellows:

$$B_i = \left\{ T_{s_i,s_j,k} \mid \frac{w_i}{w_j}, 1 \leq j \leq n, j \neq i \right\}$$

Then the farthest point in the $i$-th region always belongs to the union of the sets $A_i$ and $B_i$, i.e., $X_{i,\text{far}} \in A_i \cup B_i$.

Proof: As noted earlier, $X_{i,\text{far}}$ lies on the boundary of the $i$-th region. Consider the following two cases:

Case i) $X_{i,\text{far}}$ is on the boundary curve $V_1, V_2$ such that $V_1, V_2 \in \Omega_{s_i,s_j,k}$. If $T_{s_i,s_j,k}$ is on the boundary curve $V_1, V_2$, then according to Remark 1, $X_{i,\text{far}} \in B_i$; otherwise, since between all points on the boundary curve $V_1, V_2$ either $V_1$ or $V_2$ is the nearest point to $T_{s_i,s_j,k}$, hence according to Lemma 1 either $X_{i,\text{far}} = V_1$ or $X_{i,\text{far}} = V_2$. This means that $X_{i,\text{far}} \in A_i$.

Case ii) $X_{i,\text{far}}$ is on the boundary segment $V_3, V_4$. In this case, it follows from Lemma 2 that $X_{i,\text{far}} \in A_i$. This means that in both cases $X_{i,\text{far}} \in A_i \cup B_i$. ■
Using Theorem 1, one can develop an algorithm of complexity $O(m_i)$ to calculate the farthest point in each MW-Voronoi region, where $m_i$ is the number of vertices of the $i$-th region, $i \in \mathbf{n}$. Since typically an MW-Voronoi region does not have too many vertices, the computational complexity of calculating the farthest point is usually not very high.

C. Maxmin-vertex strategy

The Maxmin-vertex strategy selects the destination for each sensor as a point inside the corresponding MW-Voronoi region whose distance from the nearest vertex is maximized. The main idea behind this strategy is that normally for a good coverage result, none of the sensors should be too close to any of its vertices. This point will be referred to as the Maxmin-vertex centroid, and will be denoted by $\hat{O}_i$ for the $i$-th MW-Voronoi region, $i \in \mathbf{n}$. Let the distance between this point and the nearest vertex on the $i$-th region to it be represented by $\bar{r}_i$. Denote with $C(O_i, r_i)$ a circle of radius $r_i$ centered at the point $O_i$. The Maxmin-vertex circle is defined next.

**Definition 7.** The Maxmin-vertex circle of a MW-region is defined as the largest circle centered inside the region such that all of the vertices of the region are either outside the circle, or on it. This circle is, in fact, the Maxmin-vertex circle.

**Lemma 3.** Suppose the $i$-th MW-Voronoi region, $i \in \mathbf{n}$, has more than one boundary curve. If the Maxmin-vertex circle passes through exactly one vertex, say $V_{11}$, then $\hat{O}_i$ is $T_{\Omega}^{V_{11}}$ for some $\Omega \in \mathbf{\Omega}_i$; otherwise, the Maxmin-vertex circle passes through at least two vertices.

**Proof:** Let $\bar{V}_{11}$ be the nearest vertex of the $i$-th MW-Voronoi region to $\hat{O}_i$, and define:

$$\hat{u} := \min_{v \in \mathbf{V}_i - \{\bar{V}_{11}\}} \{d(\hat{O}_i, V)\}, \quad i \in \mathbf{n} \tag{6}$$

where $\mathbf{V}_i$ is the set of vertices of $i$-th MW-Voronoi region in the MW-Voronoi diagram.

![Fig. 5. An example of the Maxmin-vertex circle, when it passes through exactly one vertex.](image)

Suppose $\hat{O}_i$ and $T_{\Omega}^{V_{11}}$ are disjoint for any $\Omega \in \mathbf{\Omega}_i$. Suppose also that the Maxmin-vertex circle does not pass through any vertex other than $V_{11}$, and hence the parameter $\delta^* = (\hat{u} - \bar{r}_i)/2$ is strictly positive. There are two possible cases, as discussed below.

Case 1: $\hat{O}_i$ is inside the $i$-th MW-Voronoi region. Let $\hat{O}$ be a point on the line $\bar{V}_{11}\hat{O}_i$, but closer to $\hat{O}_i$, such that the distance between $\hat{O}_i$ and $\hat{O}$ is equal to $\delta$, where $0 < \delta \leq \delta^*$ (see Fig. 6(a)).

Case 2: $\hat{O}_i$ is on the boundary of the $i$-th MW-Voronoi region. Suppose $\hat{O}_i$ is on the curve $\epsilon$. Since $\hat{O}_i$ and $T_{\Omega}^{V_{11}}$ are distinct for any $\Omega \in \mathbf{\Omega}_i$, one can choose a point $\hat{O}$ on $\epsilon$ such that $d(\hat{O}, \bar{V}_{11}) > d(\hat{O}_i, \bar{V}_{11})$ and the distance between $\hat{O}_i$ and $\hat{O}$ is equal to $\delta$, where $0 < \delta \leq \delta^*$ (see Fig. 6(b)).

In both cases, according to the triangle inequality:

$$d(\hat{O}, V) \geq d(\hat{O}_i, V) - \delta \geq \hat{u} - \bar{r}_i, \quad \forall V \in \mathbf{V}_i - \{\bar{V}_{11}\} \tag{7}$$

From the above relation and on nothing that $\hat{u} - \bar{r}_i \geq \bar{r}_i + \delta > \bar{r}_i$ and $d(\hat{O}, \bar{V}_{11}) > d(\hat{O}_i, \bar{V}_{11})$, it can be concluded that

$$\min_{V \in \mathbf{V}_i} \{d(\hat{O}, V)\} > \bar{r}_i, \quad i \in \mathbf{n} \tag{8}$$

which contradicts the fact that $\hat{O}_i$ is the Maxmin-vertex centroid. This means that there is at least one more vertex on the Maxmin-vertex circle. ■
Lemma 4. Consider a MW-Voronoi diagram, and assume that the Maxmin-vertex circle of one region, say region $i$, $i \in n$, passes through exactly two vertices, say $V_{i1}$ and $V_{i2}$. Then $O_i$ is the intersection point of the perpendicular bisector of $V_{i1}V_{i2}$ and the boundary of the $i$-th MW-Voronoi region.

Proof: Suppose $O_i$ is not the intersection point of the perpendicular bisector of $V_{i1}V_{i2}$ and the boundary of the $i$-th MW-Voronoi region, i.e., $O_i$ is inside the $i$-th MW-Voronoi region. Define:

$$\tilde{u} := \min_{V \in V_i - \{V_{i1}, V_{i2}\}} \{d(O_i, V)\}, \quad i \in n$$  \hspace{1cm} (9)

Since $C(\tilde{O}_i, \tilde{r}_i)$ passes through exactly two vertices, thus $\delta^* = (\tilde{u} - \tilde{r}_i)/2$ is strictly positive. Let $\tilde{O}$ be a point on the perpendicular bisector of $V_{i1}V_{i2}$ and outside the triangle $V_{i1}V_{i2}O_i$, but closer to $O_i$. Denote the distance between the points $O_i$ and $\tilde{O}$ with $\delta$, where $0 < \delta \leq \delta^*$ (see Fig. 7). Using the triangle inequality, one can write:

$$d(\tilde{O}, V) \geq d(O_i, V) - \delta \geq \tilde{u} - \delta, \quad \forall V \in V_i - \{V_{i1}, V_{i2}\}$$  \hspace{1cm} (10)

From the above relation and the inequalities $\tilde{u} - \delta \geq \tilde{u} - \delta^* = \tilde{r}_i + \delta^* > \tilde{r}_i$ and $d(\tilde{O}, V_{i1}) = d(O_i, V_{i2}) > \tilde{r}_i$, it yields that:

$$\min_{V \in V_i} \{d(\tilde{O}, V)\} > \tilde{r}_i, \quad i \in n$$  \hspace{1cm} (11)

which contradicts the fact that $O_i$ is the Maxmin-vertex centroid. This completes the proof. □

Definition 8. For convenience of notation, the circle passing through two vertices $V_p$ and $V_q$ of MW-Voronoi region $i$ (with $m_i$ vertices), centered at the intersection of the perpendicular bisector of $V_pV_q$ and the curve $V_kV_l$, is denoted by $\Omega_{k,l}^{p,q}$, $k, l, p, q \in m_i := \{1, \ldots, m_i\}$. Also the circle passing through one vertex $V_r$ of MW-Voronoi region $i$, centered at $T_{r,\Omega}$, is denoted by $\Omega_{\Omega}^{r}$, for any $r \in m_i$ and $\Omega \in \Omega_i$. 

Fig. 6. The Maxmin-vertex centroid, when it is: (a) inside the MW-Voronoi region, and (b) on the boundary of the MW-Voronoi region.

Fig. 7. An illustrative figure used in the proof of Lemma 4.
Theorem 2. Suppose the i-th MW-Voronoi region has more than one boundary curve. Let $\hat{C}_i$ and $\tilde{C}_i$ be the sets of all circles $\Omega_{p,q}^{k,l}$, $\forall k, l, p, q \in \mathbf{m}_i$, and $\Omega_{i}^{k,l}$, $\forall r \in \mathbf{m}_i$, $\Omega \in \mathbf{M}_i$, respectively, whose centers are on the boundary curve of MW-Voronoi region i, and do not enclose any of the vertices of the MW-Voronoi region. Let also $\hat{C}_i$ be the set of all circumcircles of any three vertices, centered inside the MW-Voronoi region or on its boundary, which do not enclose any of the vertices of the MW-Voronoi region. Define $C_i := \hat{C}_i \cup \tilde{C}_i \cup \hat{C}_i$; then $C(O, \tilde{r}_i) \in C_i$. Moreover, for all $C(O, r) \in C_i$, $r \leq \tilde{r}_i$.

Proof: If $C(O_i, \tilde{r}_i) \notin \hat{C}_i$, then according to Lemma 3 the Minmax-vertex circle passes through at least two vertices. If it passes through exactly two vertices, say $V_1, V_2$, then according to Lemma 4, there exist $k, l \in \mathbf{m}_i$ such that $C(O, \tilde{r}_i) = \Omega_{k,l}^{i}$. Hence, in this case $C(O_i, \tilde{r}_i) \in C_i$, and from Definition 7, $\tilde{r}_i = \max_{C(O, r) \in C_i} \{r\}$. If, on the other hand, the Minmax-vertex circle passes through three or more Voronoi vertices, then it is the circumcircle of those vertices. Therefore, $C(O_i, \tilde{r}_i) \in C_i$, and again it is deduced from Definition 7 that $\tilde{r}_i = \max_{C(O, r) \in C_i} \{r\}$.

Remark 2. If the i-th MW-Voronoi region has exactly one boundary curve, then this curve is a circle and it is considered as the Maxmin-vertex circle in the Maxmin-vertex strategy.

Using the result of Theorem 2, one can develop a procedure with a complexity of order $O(m_i^3)$ to calculate the Maxmin-vertex centroid in the i-th MW-Voronoi region, where $m_i$ is the number of the vertices of the corresponding region. Since typically a MW-Voronoi region does not have too many vertices, the computational complexity for calculating the Maxmin-vertex centroid is usually not very high.

D. Minmax-vertex strategy

The idea behind the Minmax-vertex technique is that normally a sensor should not be too far from any of its Voronoi vertices when the covered area is maximized. The Minmax-vertex strategy selects the target location for each sensor as a point inside the corresponding MW-Voronoi region whose distance from the farthest vertex is minimized. This point will be referred to as the Minmax-vertex centroid, and will be denoted by $\hat{O}_i$ for the i-th region, $i \in \mathbf{n}$. Furthermore, the distance between this point and the farthest vertex from it will be represented by $\hat{r}_i$. The Maxmin-vertex circle is defined next.

Definition 9. The Minmax-vertex circle of a MW-Voronoi region is defined as the smallest circle centered inside the region such that all of the vertices of the region are either inside the circle, or on it. This circle is, in fact, $C(O_i, \hat{r}_i)$, for the i-th region.

Lemma 5. If a MW-Voronoi region has more than one boundary curve, then the corresponding Minmax-vertex circle passes through at least two vertices.

Proof: Let $\hat{V}_i$ be the farthest vertex on the boundary of MW-Voronoi region i to $\hat{O}_i$, and define:

$$\hat{\varepsilon} := \max_{V \in V_i - \{\hat{V}_i\}} \{d(\hat{O}_i, V)\}, \quad i \in \mathbf{n}$$

(12)

where $V_i$ is the set of all vertices of MW-Voronoi region i. Suppose that the Minmax-vertex circle does not pass through any vertex other than $\hat{V}_i$, and hence $\hat{\varepsilon} = (\hat{r}_i - \hat{\varepsilon})/2$ is strictly positive. There are two possible cases, as discussed below.

Case 1: $\hat{O}_i$ is inside the MW-Voronoi region. Let $\hat{O}$ be a point inside the MW-Voronoi region and on the line $\hat{V}_i \hat{O}_i$ such that the distance between $\hat{O}_i$ and $\hat{O}$ is equal to $\delta$, where $0 < \delta \leq \hat{\varepsilon}$ (see Fig. 8(a)).

Case 2: $\hat{O}_i$ is on boundary of the MW-Voronoi region. Suppose $\hat{O}_i$ is on the curve $\epsilon$. Let $\hat{O}$ be a point on $\epsilon$ or in the i-th MW-Voronoi region such that $d(\hat{O}, \hat{V}_i) < d(\hat{O}_i, \hat{V}_i)$ and the distance between $\hat{O}_i$ and $\hat{O}$ is equal to $\delta$, where $0 < \delta \leq \hat{\varepsilon}$ (see Fig. 8(b)).

In both cases, according to the triangle inequality:

$$d(\hat{O}, V) \leq d(\hat{O}_i, V) + \delta \leq \hat{\varepsilon} + \delta, \quad \forall V \in V_i - \{\hat{V}_i\}$$

(13)

From the above relation and on noting that $\hat{\varepsilon} + \delta \leq \hat{r}_i - \varepsilon_i$ and $d(\hat{O}, \hat{V}_i) < d(\hat{O}_i, \hat{V}_i)$, it can be concluded that

$$\max_{V \in V_i} \{d(\hat{O}, V)\} < \hat{r}_i$$

(14)

which contradicts the fact that $\hat{O}_i$ is the Minmax-vertex centroid. This means that there is at least one more vertex on the Minmax-vertex circle.

Lemma 6. Consider a MW-Voronoi diagram, and assume that the Minmax-vertex circle of one region, say region i, $i \in \mathbf{n}$, passes through exactly two vertices, say $\hat{V}_{i1}$ and $\hat{V}_{i2}$. Then $\hat{O}_i$ is the intersection point of the perpendicular bisector of $\hat{V}_{i1}\hat{V}_{i2}$ and the boundary of the i-th MW-Voronoi region.
Proof: Suppose \( \hat{O}_i \) is not the intersection point of the perpendicular bisector of \( \hat{V}_{1i}\hat{V}_{2i} \) and the boundary of the \( i \)-th MW-Voronoi region, i.e., \( \hat{O}_i \) is inside the region. Define:

\[
\hat{z} := \max_{V \in \mathbf{V}_i - \{\hat{V}_{1i}, \hat{V}_{2i}\}} \{d(\hat{O}_i, V)\}, \quad i \in \mathbf{n}
\]  

(15)

Since \( C(\hat{O}_i, \hat{r}_i) \) passes through exactly two vertices, thus \( \delta^* = (\hat{r}_i - \hat{z})/2 \) is positive. Let \( \check{O} \) be a point on the perpendicular bisector of \( \hat{V}_{1i}\hat{V}_{2i} \) and inside the triangle \( \hat{V}_{1i}\hat{V}_{2i}\hat{O}_i \), but closer to \( \hat{O}_i \), the distance between the points \( \hat{O}_i \) and \( \check{O} \) is equal to \( \delta \), where \( 0 < \delta \leq \delta^* \) (see Fig. 9). Using the triangle inequality, one can write:

\[
d(\check{O}, V) \leq d(\hat{O}_i, V) + \delta \leq \hat{z} + \delta, \quad \forall V \in \mathbf{V}_i - \{\hat{V}_{1i}, \hat{V}_{2i}\}
\]  

(16)

From the above relation and the inequalities \( \hat{z} + \delta \leq \hat{z} + \delta^* = \hat{r}_i - \delta^* < \hat{r}_i \) and \( d(\hat{O}, \hat{V}_{1i}) = d(\check{O}, \hat{V}_{1i}) < \hat{r}_i \), it yields that:

\[
\max_{V \in \mathbf{V}_i} \{d(\check{O}, V)\} < \hat{r}_i, \quad i \in \mathbf{n}
\]  

(17)

which contradicts the fact that \( \hat{O}_i \) is the Minmax-vertex centroid.

\[
\blacksquare
\]

Theorem 3. Let \( \mathbf{W}_i \) be the set of all circles \( \Omega_{p,q}^{k,l} \) \( \forall k, l, p, q \in \mathbf{m}_i \), whose centers are on the boundary of MW-Voronoi region \( i \), and all the vertices of the region are either inside or on them. Let also \( \tilde{\mathbf{W}}_i \) be the set of all circumcircles of any three vertices, centered inside or on MW-Voronoi region \( i \), with all the vertices of the region either inside or on them. Define \( \mathbf{W}_i := \mathbf{W}_i \cup \tilde{\mathbf{W}}_i \); then \( C(\hat{O}_i, \hat{r}_i) \in \mathbf{W}_i \). Moreover, for all \( C(\mathcal{O}, r) \in \mathbf{W}_i \), the inequality \( r \geq \hat{r}_i \) holds.

Proof: According to Lemma 5, the Minmax-vertex circle passes through at least two Voronoi vertices. If it passes through exactly two Voronoi vertices, say \( \hat{V}_1, \hat{V}_2 \), then according to Lemma 6, there exist \( k, l \in \mathbf{m}_i \), such that \( C(\hat{O}_i, \hat{r}_i) = \Omega_{p,q}^{k,l} \). Hence, in this case \( C(\hat{O}_i, \hat{r}_i) \in \mathbf{W}_i \), and from Definition 9, \( \hat{r}_i = \min_{C(\mathcal{O}, r) \in \mathbf{W}_i} \{r\} \). If, on the other hand, the Minmax-vertex circle
Definition 11. \( k > d \)

Remark 3. If the \( i \)-th MW-Voronoi region has exactly one boundary curve, then this curve is a circle and it is considered as the Minmax-vertex circle in the Minmax-vertex strategy.

Using the result of Theorem 3, one can develop a procedure with a complexity of order \( O(m_i^2) \) to calculate the Minmax-vertex centroid in the \( i \)-th MW-Voronoi region, where \( m_i \) is the number of the vertices of the corresponding region. Since typically a Voronoi region does not have too many vertices, the computational complexity for calculating the Minmax-vertex centroid is normally not very high.

E. Minmax-curve strategy

The idea behind the Minmax-curve technique is that normally a sensor should not be too far from any of its Voronoi curves when the covered area is maximized. The Minmax-curve strategy selects the target location for each sensor as a point inside the corresponding MW-Voronoi region whose distance from the farthest curve is minimized. This point will be referred to as the Minmax-curve centroid, and will be denoted by \( O_i \) for the \( i \)-th region, \( i \in n \). Furthermore, the distance between this point and the farthest curve from it will be represented by \( d_i \). The Minmax-curve circle is defined next.

Definition 10. The Minmax-curve circle of the \( i \)-th MW-Voronoi region is the smallest circle centered inside or on that region, intersecting or touching the region’s all curves (or their extensions). This circle is, in fact, \( C(O_i, d_i) \), and is not necessarily unique (this issue will be addressed later).

Some preliminary results will be presented in the sequel, which will be used in the Minmax-curve and Maxmin-curve strategies.

Lemma 7. Consider two points \( A \) and \( B \) in a 2-D plane, and let the distance between \( A \) and \( B \) be \( d \). The loci of points like \( E \) such that \( d(E, A) - d(E, B) = k \) is:

i) The perpendicular bisector of segment \( AB \), for \( k = 0 \).
ii) A branch of a hyperbola, for \( 0 < k < d \).
iii) The extension of the segment \( AB \) from \( B \), for \( k = d \).
iv) The empty set, for \( k > d \).

Proof: The proof is obvious according to triangle inequality and definition of perpendicular bisector and hyperbola.

Lemma 8. Consider two points \( A \) and \( B \) in a 2-D plane, and let the distance between \( A \) and \( B \) be \( d \). The loci of points like \( E \) such that \( d(E, A) + d(E, B) = k \) is:

i) An empty set, for \( k < d \).
ii) Segment \( AB \), for \( k = d \).
iii) An ellipse, for \( k > d \).

Proof: According to definition of ellipse and triangle inequality the proof is straightforward.

Definition 11. The bisector of two curves \( \epsilon_1 \) and \( \epsilon_2 \) is defined as the loci of points like \( E \) that its distances to \( \epsilon_1 \) is equal to its distance to \( \epsilon_2 \). The bisector of curves \( \epsilon_1 \) and \( \epsilon_2 \) is denoted by \( \Gamma_{\epsilon_1, \epsilon_2} \).

Definition 12. For convenience of notation, for any two curves \( \epsilon_1 \) and \( \epsilon_2 \), the sets \( \Psi_{\epsilon_1, \epsilon_2}^{\max} \) and \( \Psi_{\epsilon_1, \epsilon_2}^{\min} \) is defined as follows:

\[
\Psi_{\epsilon_1, \epsilon_2}^{\min} = \{ X \in \Gamma_{\epsilon_1, \epsilon_2} : \exists \delta > 0 : \forall Y \in \Gamma_{\epsilon_1, \epsilon_2}, |Y - X| \leq \delta \Rightarrow d(X, \epsilon_1) \leq d(Y, \epsilon_1) \} \tag{18}
\]

\[
\Psi_{\epsilon_1, \epsilon_2}^{\max} = \{ X \in \Gamma_{\epsilon_1, \epsilon_2} : \exists \delta > 0 : \forall Y \in \Gamma_{\epsilon_1, \epsilon_2}, |Y - X| \leq \delta \Rightarrow d(X, \epsilon_1) \geq d(Y, \epsilon_1) \} \tag{19}
\]

Lemma 9. Consider two circles \( \Omega_1(O_1, r_1) \) and \( \Omega_2(O_2, r_2) \). The bisector of \( \Omega_1 \) and \( \Omega_2 \) is:

i) a branch of a hyperbola or the perpendicular bisector of \( O_1O_2 \), if \( \Omega_2 \) is outside of \( \Omega_1 \).
ii) an ellipse if \( \Omega_2 \) is inside \( \Omega_1 \).
iii) the union of a branch of a hyperbola or the perpendicular bisector of \( O_1O_2 \) and an ellipse, if \( \Omega_1 \) intersects \( \Omega_2 \).

Proof:

i) Consider two circles \( \Omega_1(O_1, r_1) \) and \( \Omega_2(O_2, r_2) \) where \( \Omega_2 \) is outside of \( \Omega_1 \). Let \( E \) be a point on the plane, such that \( d(E, \Omega_1) = d(E, \Omega_2) = d \) (see, e.g. Fig 10). Because of \( EO_1 = d + r_1 \) and \( EO_2 = d + r_2 \), one can easily conclude:

\[
EO_1 - EO_2 = r_1 - r_2 = cte \tag{20}
\]
According to (20) and parts (i) and (ii) of Lemma 7 the proof is complete.

\[
EO_1 + EO_2 = r_1 + r_2 = \text{cte} \tag{21}
\]

According to (21) and part (iii) of Lemma 8 the proof is complete.

\[
EO_1 - EO_2 = r_1 - r_2 = \text{cte} \tag{22}
\]

\[
FO_1 + FO_2 = r_1 + r_2 = \text{cte} \tag{23}
\]

According to (22), (23), parts (i) and (ii) of Lemma 7 and part (iii) of Lemma 8 the proof is complete.

**Lemma 10.** Consider circle \( \Omega(O, r) \) and line \( \Delta \). The bisector of \( \Omega \) and \( \Delta \) is:

\begin{enumerate}
\item[i)] a parabola, if line \( \Delta \) does not intersect circle \( \Omega \)
\item[ii)] the union of two parabolas, if line \( \Delta \) intersects circle \( \Omega \)
\end{enumerate}

*Proof:* The proof is straightforward and is omitted here.

**Lemma 11.** Consider two points \( A, B \) and a curve \( \epsilon \), which is either circle \( \Omega(O, r) \) or line \( \Delta \). Let the distance between \( A \) and \( \epsilon \) be given by \( \sigma \), and that between \( B \) and \( \epsilon \) by \( \rho \). Let also the length of the segment \( AB \) be denoted by \( \xi \). Then:

\[
\sigma - \xi \leq \rho \leq \sigma + \xi \tag{24}
\]
Proof: Let $E$ and $F$ be two points on $\epsilon$, such that $AE \perp \epsilon$ and $BF \perp \epsilon$. Let $d(A, F) = d_1$ and $d(B, E) = d_2$.

If $\epsilon$ is Line $\Delta$ (see Fig. 15), then according to the triangle inequality:

\begin{align}
\rho + \xi & \geq d_1 \geq \sigma & (25) \\
\sigma + \xi & \geq d_2 \geq \rho & (26)
\end{align}

Equations (25) and (26) complete the proof.

If $\epsilon$ is circle $\Omega(O, r)$ (see Fig. 16), then according to the triangle inequality, $OA + AB \geq OB$ and $OB + AB \geq OA$, and because of $OA = r + \sigma$ and $OB = r + \rho$ one can conclude:

\begin{align}
\sigma + \xi & \geq \rho & (27) \\
\rho + \xi & \geq \sigma & (28)
\end{align}
The inequality (24) follows directly from the above inequalities, and this completes the proof.

**Lemma 12.** If a MW-Voronoi region has more than one boundary curve, then the corresponding Minmax-curve circle is tangent to at least two of the boundary curves (or their extensions).

**Proof:** Suppose the \(i\)-th MW-Voronoi region has more than one boundary curve. Let \(\epsilon_{i1}\) be the farthest boundary curve from the Minmax-curve centroid of the \(i\)-th MW-Voronoi region. It is obvious that \(\hat{r}_{i} \) is equal to the distance between \(\hat{O}_{i}\) and \(\epsilon_{i1}\), also denoted by \(d(\hat{O}_{i}, \epsilon_{i1})\). Thus, \(C(\hat{O}_{i}, \hat{r}_{i})\) is tangent to \(\epsilon_{i1}\) (or its extension). Define:

\[
\hat{v} := \max_{\epsilon_{i} \in \epsilon_{i1}-\{\epsilon_{i1}\}} \left\{ d(\hat{O}_{i}, \epsilon) \right\}
\]

(29)

where \(\epsilon_{i}\) represents the set of all boundary curves of the \(i\)-th MW-Voronoi region, and suppose that the Minmax-curve circle is not tangent to any other edge, implying that \(\delta^{*} = (\hat{r}_{i} - \hat{v})/2\) is strictly positive. Let \(M\) be a point on \(\epsilon_{i1}\) or its extension, such that \(M\hat{O}_{i} \perp \epsilon_{i1}\). Let also \(\hat{O}\) be a point on \(M\hat{O}\) such that \(\hat{O}_{i}\hat{O} = \delta\), where \(0 < \delta < \delta^{*}\) (see, e.g. Fig. 17). According to Lemma 11:

\[
d(\hat{O}, \epsilon) \leq d(\hat{O}_{i}, \epsilon) + \delta \leq \hat{v} + \delta, \forall \epsilon \in \epsilon_{i} - \{\epsilon_{i1}\}
\]

(30)

From inequality (30) and the relation \(\hat{v} + \delta < \hat{r}_{i} - \delta < \hat{r}_{i}\), and \(d(\hat{O}, \epsilon_{i1}) < d(\hat{O}_{i}, \epsilon_{i1})\), one can conclude that:

\[
\max_{\epsilon \in \epsilon_{i1}} \left\{ d(\hat{O}, \epsilon) \right\} < \hat{r}_{i}
\]

(31)

which contradicts the fact that \(\hat{O}_{i}\) is the Minmax-curve centroid.

**Lemma 13.** Suppose the \(i\)-th MW-Voronoi region has more than two boundary curves and its Minmax-curve circle is tangent to exactly two curves, say \(\epsilon_{i1}\) and \(\epsilon_{i2}\). Then \(\epsilon_{i1}\) and \(\epsilon_{i2}\) are parallel, or \(\hat{O}_{i} \in \Psi_{\epsilon_{i1}, \epsilon_{i2}}^{min}\), or \(\hat{O}_{i}\) is the intersection of the bisector of \(\epsilon_{i1}\), \(\epsilon_{i2}\), and one boundary curve of the region; otherwise, it is tangent to at least three Voronoi curves (or their extensions).

**Proof:** Suppose the \(i\)-th MW-Voronoi region has more than two boundary curves and its Minmax-curve circle is tangent to exactly two non-parallel Voronoi curves \(\epsilon_{i1}\) and \(\epsilon_{i2}\), but \(\hat{O}_{i}\) is not the intersection of the bisector of \(\epsilon_{i1}\), \(\epsilon_{i2}\), and any curve of the region; i.e., \(\hat{O}_{i}\) is inside the region. Define:

\[
\hat{v} := \max_{\epsilon \in \epsilon_{i1}, \epsilon_{i2}} \left\{ d(\hat{O}, \epsilon) \right\}
\]

(32)
Fig. 17. An illustrative figure used in the proof of Lemma 12

Since $C(\hat{O}_i, \hat{r}_i)$ is tangent to exactly two curves, the term $\delta^* = (\hat{r}_i - \hat{v})/2$ is strictly positive. If $\hat{O}_i$ is not in $\psi^\text{min}_{\hat{r}_1, \hat{r}_2}$, then one can choose a point inside the Minmax-circle and on the bisector of $\hat{r}_1$ and $\hat{r}_2$, say $\hat{O}$, such that $\hat{O}_i \hat{O} = \delta$, where $0 < \delta \leq \delta^*$ and $d(\hat{O}, \hat{r}_1) = d(\hat{O}, \hat{r}_2) < \hat{r}_i$, (e.g., see Fig. 18).

According to Lemma 11:

![Fig. 18. An illustrative figure used in the proof of Lemma 13](image_url)

$$d(\hat{O}, \epsilon) \leq d(\hat{O}_i, \epsilon) + \delta \leq \hat{v} + \delta,$$ \quad $\forall \epsilon \in \epsilon_i - \{\hat{r}_1, \hat{r}_2\}$ \quad (33)

It results from inequality (33) and the relations $\hat{v} + \delta \leq \hat{r}_i - \delta < \hat{r}_i$ and $d(\hat{O}, \hat{r}_1) = d(\hat{O}, \hat{r}_2) < \hat{r}_i$, that:

$$\max_{\epsilon \in \epsilon_i} \left\{ d(\hat{O}, \epsilon) \right\} < \hat{r}_i$$ \quad (34)

which contradicts the fact that $\hat{O}_i$ is the Minmax-curve centroid. On the other hand, if the Minmax-curve circle is not touching exactly two Voronoi curves (or their extensions), then according to Lemma 12 it is tangent to at least three Voronoi curves (or their extensions). This completes the proof. \hfill \blacksquare

**Definition 13.** For convenience of notation, the circle touching two curves $\epsilon_g$ and $\epsilon_h$ of MW-Voronoi region $i$, centered at the intersection of the bisector $\epsilon_g$ and $\epsilon_h$ and the curve $\epsilon_k$, is denoted by $\Omega^k_{g,h}$, for any $k, g, h \in \epsilon_i := \{1, \ldots, \epsilon_i\}$. Also the circle touching two curves $\epsilon_r$ and $\epsilon_s$ of MW-Voronoi region $i$, centered at the point $A \in \psi^\text{min}_{\epsilon_r, \epsilon_s}$, is denoted by $\Omega^A_{r,s}$ for any $r, s \in \epsilon_i$.

**Definition 14.** Suppose $\epsilon_1$ and $\epsilon_2$ be the circular arcs of circles $C_1$ and $C_2$ respectively. The curves $\epsilon_1$ and $\epsilon_2$ are called parallel if circles $C_1$ and $C_2$ be concentric.

**Lemma 14.** If a Minmax-curve circle is tangent to two parallel curves (or their extensions), then there may also be other Minmax-curve circles, all of which would be tangent to these parallel curves (or extensions), as well.

**Proof:** Suppose one Minmax-curve circle, say $C_1$, is tangent to two parallel curves, say $\epsilon_1$ and $\epsilon_2$, but there exists another Minmax-curve circle, say $C_2$, that is not tangent to $\epsilon_1$ or $\epsilon_2$. Let the distance between $\epsilon_1$ and $\epsilon_2$ be denoted by
$d(\epsilon_1, \epsilon_2)$. It is obvious that the radius of the circle $C_1$ is equal to $\frac{d(\epsilon_1, \epsilon_2)}{2}$, and that of the circle $C_2$ is greater than $\frac{d(\epsilon_1, \epsilon_2)}{2}$. This contradicts the fact that $C_2$ is a Minmax-curve circle. \hfill \blacksquare

Remark 4. When MW-Voronoi region has more than two boundary curves and all Minmax-curve circles are tangent to two parallel curves (or their extensions), then one of possibly multiple such circles that are tangent to three or more curves is arbitrarily considered as the Minmax-curve circle.

Remark 5. When the Minmax-curve circle is not unique, then one of them is arbitrarily considered as the Minmax-curve circle in the Maxmin-curve strategy.

Theorem 4. Suppose the $i$-th MW-Voronoi region has more than two boundary curves. Let $\hat{D}_i$ and $\hat{\hat{D}}_i$ be the sets of all circles $\Omega_{g,h}^i$, $\forall k, g, h \in e_i$, and $\Omega_{r,s}^{A_{min}}$, $\forall r, s \in e_i$, $A \in \Psi_{\epsilon_1, \epsilon_2}$, such that: (i) their centers lie inside or on the MW-Voronoi region, and (ii) they intersect or are tangent to all of the boundary curves of the region (or their extensions). Let also $\hat{\hat{D}}_i$ be the set of all circles such that: (i) they are tangent to at least three boundary curves of a MW-Voronoi region (or their extensions); (ii) their centers lie inside or on the region, and (iii) they intersect or are tangent to all of the boundary curves of the MW-Voronoi region (or their extensions). Define $D_i := \hat{D}_i \cup \hat{\hat{D}}_i \cup \hat{\mathbf{D}}_i$; then the Minmax-curve circle belongs to $D_i$, and also it is the smallest circle in this set.

Proof: The proof follows directly from Lemmas 13 and 14, Remarks 4 and 5, and Definitions 10 and 13. \hfill \blacksquare

Remark 6. If the MW-Voronoi region has exactly one boundary curve, then this curve is a circle and it is considered as the Minmax-curve circle, and if it has exactly two boundary curves, then according to Lemma 12 the Minmax-curve circle is tangent to both curves.

Using the result of previous theorem and remarks, one can develop a procedure with a complexity of order $O(m_i \hat{\gamma})$ to calculate the Minmax-curve centroid in the $i$-th MW-Voronoi region, where $m_i$ is the number of the boundary curves of the corresponding region. Since typically a MW-Voronoi region does not have “too many” boundary curves, the computational complexity for calculating the Minmax-curve centroid is normally not very high.

F. The Maxmin-Curve Strategy

The main idea behind this strategy is that normally a sensor should not be too close to any of its Voronoi curves when the covered area is maximized. The Maxmin-curve strategy selects the target location for each sensor as a point inside the corresponding MW-Voronoi region whose distance from the nearest curve is maximized. This point will be referred to as the Maxmin-curve centroid, and will be denoted by $\hat{O}_i$ for the $i$-th region, $i \in \mathbf{n}$. Furthermore, the distance between this point and the nearest curve from it will be represented by $\hat{r}_i$. The Maxmin-curve circle is defined next.

Definition 15. The Maxmin-curve circle of a MW-Voronoi region is the largest circle inside the MW-Voronoi region. This circle is, in fact, $C(\hat{O}_i, \hat{r}_i)$, for the $i$-th region.

Lemma 15. If a MW-Voronoi region has more than one boundary curve, then the corresponding Maxmin-curve circle is tangent to at least two of the curves.

Proof: Let $\hat{\epsilon}_{i1}$ be the nearest boundary curve to the Maxmin-curve centroid of the $i$-th MW-Voronoi region. It is obvious that $\hat{r}_i$ is equal to the distance between $\hat{O}_i$ and $\hat{\epsilon}_{i1}$, also denoted by $d(\hat{O}_i, \hat{\epsilon}_{i1})$; thus, $C(\hat{O}_i, \hat{r}_i)$ is tangent to $\hat{\epsilon}_{i1}$. Define:

$$\hat{w} = \min_{\epsilon \in \epsilon_1 \setminus \{\hat{\epsilon}_{i1}\}} \left\{ d(\hat{O}_i, \epsilon) \right\} \tag{35}$$

where $\epsilon_1$ represents the set of all boundary curves of the $i$-th MW-Voronoi region, and suppose that the Maxmin-curve circle is not tangent to any other boundary curve, implying that $\delta^* = (\hat{w} - \hat{r}_i)/2$ is strictly positive. Let $M$ be a point on $\hat{\epsilon}_{i1}$, such that $M\hat{O}_i \perp \hat{\epsilon}_{i1}$. Let also $\hat{\hat{O}}$ be a point on the extension of $M\hat{O}_i$ such that $\hat{O}_i\hat{\hat{O}} = \delta$, where $0 < \delta \leq \delta^*$ (see, e.g. Fig. 19). According to Lemma 11:

$$d(\hat{O}_i, \epsilon) \geq d(\hat{O}_i, \epsilon) - \delta \geq \hat{w} - \delta, \ \forall \epsilon \in \epsilon_1 \setminus \{\hat{\epsilon}_{i1}\} \tag{36}$$

From inequality (36) and relations $\hat{w} - \delta \geq \hat{r}_i + \delta > \hat{r}_i$ and $d(\hat{O}_i, \hat{\epsilon}_{i1}) > d(\hat{O}_i, \hat{\epsilon}_{i1})$, one can conclude that:

$$\min_{\epsilon \in \epsilon_1 \setminus \{\hat{\epsilon}_{i1}\}} \left\{ d(\hat{O}_i, \epsilon) \right\} > \hat{r}_i \tag{37}$$
which contradicts the fact that $\hat{O}_i$ is the Maxmin-curve centroid. This completes the proof.

Lemma 16. Suppose the $i$-th MW-Voronoi region has more than two boundary curves and its Maxmin-curve circle is tangent to exactly two curves, say $\hat{\epsilon}_{i1}, \hat{\epsilon}_{i2}$. Then these two curves are either parallel or $\hat{O}_i \in \Psi_{\hat{\epsilon}_{i1}, \hat{\epsilon}_{i2}}^{\max}$, otherwise, it is tangent to at least three Voronoi curves.

Proof: Suppose the $i$-th MW-Voronoi region has more than two boundary curves and its Maxmin-curve circle is tangent to exactly two curves, say $\hat{\epsilon}_{i1}$ and $\hat{\epsilon}_{i2}$, but these two curves are not parallel. Define:

$$\bar{w} := \min_{\epsilon \in \epsilon_i - \{\hat{\epsilon}_{i1}, \hat{\epsilon}_{i2}\}} \left\{ d(\hat{O}_i, \epsilon) \right\}$$ (38)

Since $C(\hat{O}_i, \bar{r}_i)$ is tangent to exactly two boundary curves, the term $\delta^* = (\bar{w} - \bar{r})/2$ is strictly positive. If $\hat{O}_i \notin \Psi_{\hat{\epsilon}_{i1}, \hat{\epsilon}_{i2}}^{\max}$, then one can choose a point inside the $i$-th MW-Voronoi region and on the bisector of $\hat{\epsilon}_{i1}$ and $\hat{\epsilon}_{i2}$, say $\tilde{O}$, such that $\tilde{O}O = \delta$, where $0 < \delta \leq \delta^*$ and $d(\tilde{O}, \hat{\epsilon}_{i1}) = d(\tilde{O}, \hat{\epsilon}_{i2}) > \bar{r}_i$ (e.g., see Fig. 20).

According to Lemma 11:

$$d(\hat{O}, \epsilon) \geq d(\hat{O}_i, \epsilon) - \delta \geq \bar{w} - \delta, \quad \forall \epsilon \in \epsilon_i - \{\hat{\epsilon}_{i1}, \hat{\epsilon}_{i2}\}$$ (39)

It results from inequality (39) and the relations $\bar{w} - \delta \geq \bar{r}_i + \delta > \bar{r}_i$ and $d(\hat{O}, \hat{\epsilon}_{11}) = d(\hat{O}, \hat{\epsilon}_{i2}) > \bar{r}_i$, that:

$$\min_{\epsilon \in \epsilon_i} \left\{ d(\hat{O}, \epsilon) \right\} > \bar{r}_i$$ (40)

which contradicts the fact that $\hat{O}_i$ is the Maxmin-curve centroid. On the other hand, if the Maxmin-curve circle is not touching exactly two curves, according to Lemma 15 it is tangent to at least three curves. This completes the proof.

Definition 16. For convenience of notation, the circle touching two curves $\epsilon_r$ and $\epsilon_s$ of MW-Voronoi region $i$, centered at the point $A \in \Psi_{\epsilon_r, \epsilon_s}^{\max}$, is denoted by $\Omega_{r,s}^{A,\max}$ for any $r, s \in \epsilon_i$. 

Fig. 19. An illustrative figure used in the proof of Lemma 15

Fig. 20. An illustrative figure used in the proof of Lemma 16
Lemma 17. If a Maxmin-curve circle is tangent to two parallel edges, then there may also be other Maxmin-curve circles, all of which would be tangent to these parallel curves.

Proof: Suppose one Maxmin-curve circle, say $C_1$, is tangent to two parallel edges, say $e_1$ and $e_2$, but there exists another Maxmin-curve circle, say $C_2$, that is not tangent to $e_1$ or $e_2$. It is obvious that the radius of circle $C_1$ is equal to $\frac{d(e_1,e_2)}{2}$, and that of the circle $C_2$ is less than $\frac{d(e_1,e_2)}{2}$. This contradicts the fact that $C_2$ is a Maxmin-curve circle. ■

Remark 7. When MW-Voronoi region has more than two boundary curves and all Maxmin-curve circles are tangent to two parallel edges, then one of possibly multiple such circles that are tangent to three or more edges is arbitrarily considered as the Maxmin-curve circle.

Remark 8. When the Maxmin-curve circle is not unique, then one of them is arbitrarily considered as the Maxmin-curve circle in the Maxmin-curve strategy.

Theorem 5. Suppose the $i$-th MW-Voronoi region has more than two boundary curves. Let $\hat{Z}_i$ be the set of all circles $\Omega_{r,s}^{A,max}, \forall r,s \in \mathbb{E}_i, A \in \Psi_{r,s}^{max}$ such that are inside the region. Let also $\tilde{Z}_i$ be the set of all circles which: (i) are tangent to at least three curves of a MW-Voronoi region, and (ii) are inside the region. Define $Z_i := \hat{Z}_i \cup \tilde{Z}_i$; then the Maxmin-curve circle belongs to $Z_i$, and also it is the largest circle in this set.

Proof: The proof follows directly from Lemmas 16 and 17, Remarks 7 and 8 and Definitions 15 and 16. ■

Remark 9. If the MW-Voronoi region has exactly one boundary curve, then this curve is a circle and by itself it is the Maxmin-curve circle; and if it has exactly two boundary curves, then according to Lemma 15 the Maxmin-curve circle is tangent to both curves.

G. The Minmax-point Strategy

The idea behind the Minmax-point technique is that normally a sensor should not be too far from any point of its MW-Voronoi region when the covered area is maximized. The Minmax-point strategy selects the target location for each sensor as a point inside the corresponding MW-Voronoi region whose distance from the farthest point of the region is minimized. This point will be referred to as the Minmax-point centroid, and will be denoted by $\hat{r}_i$ for the $i$-th region, $i \in \mathbb{n}$. Furthermore, the distance between this point and the farthest point from it in the region will be represented by $\hat{r}_i r$. The Minmax-point circle is defined next.

Definition 17. The Minmax-point circle of a MW-Voronoi region is defined as the smallest circle centered inside the region such that all points of the region are either inside the circle, or on it. This circle is, in fact, $C(O_i, \hat{r}_i)$, for the $i$-th region, $i \in \mathbb{n}$.

Let the boundary curve $\epsilon$ be a segment of the circle $\Omega(O,r) \in \Omega_i$. If $\hat{r} \geq 180$, then $\hat{r} \geq r$.

Proof: The proof is straightforward, and is omitted here. ■

Lemma 19. Let the boundary curve $\epsilon$ be a segment of the circle $\Omega(O,r) \in \Omega_i$ and $V_1$ and $V_2$ be the end points of $\epsilon$. Let also $V$ be a point inside the $i$-th MW-Voronoi region and $\hat{r} < 180$. If $VT_{\hat{r}}^V < r$, then between all points on $\epsilon$, the farthest one from $V$ is either $V_1$ or $V_2$.

Proof: Let the intersection of $V_1 O$ and $\hat{V} O$ by $\hat{V}$ be denoted by $\hat{V}_1$ and $\hat{V}_2$ respectively. If $V$ is inside the sectors $\hat{V}_1 O V_2$, $V_2 O V_1$ or $V_1 O V_2$, then $T_{\hat{r}}^\epsilon$ is on the $\hat{V}_1 V_2$, $V_2 V_1$, or $V_1 V_2$, respectively, and therefore according to Lemma 1 either $V_1$ or $V_2$ is the farthest point to $\hat{V}$. If $V$ is inside the sector $V_2 O V_1$, then $VT_{\hat{r}}^V = r + VO \geq r$ (e.g., see Fig 22). This completes the proof. ■

Lemma 20. Let the boundary curve $\epsilon$ be a segment of the circle $\Omega(O,r) \in \Omega_i$ and $V_1$ and $V_2$ be the end points of $\epsilon$. Let also $V$ be a pint inside the $i$-th MW-Voronoi region. Between all points on $\epsilon$, only $V_1$ or $V_2$ can be the farthest point from $V$.

Proof: If $T_{\hat{r}}^\epsilon$ is not be on $\epsilon$, then according to Lemma 1 between all points on $\epsilon$, the farthest point from $V$ is either $V_1$ or $V_2$. Suppose $T_{\hat{r}}^\epsilon$ is on $\epsilon$ and $T_{\hat{r}}^\epsilon \notin V_1, V_2$, and the intersection of extension of $VT^\epsilon_{\hat{r}}$ and another boundary curve is denoted by $P$ (e.g., see Fig 22). From $VP > VT_{\hat{r}}^V$ one can easily conclude that between all points on $\epsilon$, only $V_1$ or $V_2$ can be the farthest point from $V$. ■

Lemma 21. If $\hat{r} = \min_{C(O,r) \in \Omega_i} \{r\}$, then $\hat{r} \leq \hat{r}$. ■
Proof: According to Definition 17 and because the \( i \)-th MW-Voronoi region is a subset of \( R_i = \bigcap_{\Omega \in \Omega_i} \{\Omega\} \), proof is straightforward.

**Theorem 6.** The Minmax-point circle of an MW-Voronoi region is either the Minmax-vertex circle of that region, or is in set \( \Omega_i \).

**Proof:** The proof follows directly from Lemmas 18, 19, 20 and 21.

Using the result of Theorems 3 and 6, one can develop a procedure with a complexity of order \( O(m_i^3) \) to calculate the Minmax-point centroid in the \( i \)-th MW-Voronoi region, where \( m_i \) is the number of vertices of the \( i \)-th region, \( i \in \mathbf{n} \). Since typically an MW-Voronoi region does not have too many vertices, the computational complexity for calculating the Minmax-point centroid is usually not very high.

**H. The Curtex Strategy**

By combination of the mentioned strategies one can present a new strategy. Although in the combination of the strategies obtaining the destination point is needed more calculation, it may result in better coverage. For example Curtex strategy is a combination of Minmax-vertex and Maxmin-curve strategies. In this algorithm, in each round every sensor chooses two points as its new location, one point according to Minmax-vertex strategy and the other one according to Maxmin-curve. Then one of these two points that provides better coverage is selected as the target location of the sensor. The simulation results demonstrate that the algorithm outperforms all other ones in terms of coverage.

**Remark 10.** In some cases although the moving direction of a sensor toward its assigned location is correct, but since the assigned location might be too far, the local coverage might not be increased. Therefore, as proposed in [23], the sensor may select the midpoint or 3/4 point between its current location and its next assigned location, if the new location increases its local coverage.

**Remark 11.** In order to prevent sensors oscillatory movements, the sensors check their new movement direction. If it is not in the opposite direction of the previous movement, they move to the target location; otherwise, they do not move [23].
IV. SIMULATION RESULTS

In this section, the eight algorithms proposed in Section III are applied to a flat space of size $50m \times 50m$. It is to be noted that the results presented in this section for field coverage are all the average values obtained by using 20 random initial deployments for the sensors. Furthermore, while the horizontal axis in all figures in this section represents a discrete parameter, the graphs are represented as continuous curves for clarity.

Assume first there are 27 sensors: 15 with a sensing radius of 6m, 6 with a sensing radius of 5m, 3 with a sensing radius of 7m, and 3 with a sensing radius of 9m. Moreover, the communication range of each sensor is assumed to be $10/3$ times greater than its sensing range; e.g., a sensor with a sensing range of 6m has a communication range of 20m. The coverage factor (defined as the ratio of the covered area to the overall area) of the sensors in each round is depicted in Fig. 23 for the algorithms proposed in this paper. It can be observed from this figure that all eight algorithms result in a satisfactory coverage level of the target field in the first few rounds of the corresponding algorithms. It can also be observed that for this example the Curtex algorithm performs better than the other algorithms as far as the coverage is concerned.

It is desired now to compare the performance of the proposed algorithms in terms of the number of deployed sensors $n$. To this end, consider three more set-ups: $n=18$, 36, and 45, in addition to $n=27$ discussed above. Let the changes in the number of identical sensors in the new setups be proportional to the changes in the total number of sensors (e.g., for the case of $n=18$ there will be 10 sensors with a sensing radius of 6m, 4 with a sensing radius of 5m, 2 with a sensing radius of 7m, and 2 with a sensing radius of 9m). Fig. 24 provides the coverage results for different number of sensors. It shows that although the WVB algorithm leads to satisfactory field coverage when there are a relatively small number of sensors, it is outperformed by the other algorithms as the number of sensors increases. Another important factor in the performance evaluation of different algorithms is the deployment speed in reaching the desired coverage level for the target field. Notice that the sensor deployment time in each round is almost equal for all algorithms. Hence, to compare the deployment speed, it suffices to check the number of rounds it takes for the sensors to provide a prescribed coverage level. It is shown in Fig. 25 that in all eight algorithms the number of rounds (required to meet a certain termination condition) increases by increasing the number of sensors to a certain value, and then starts to decrease by adding more sensors. This is due mainly to the fact that when there are a small number of sensors in the target field, the MW-Voronoi regions are large in comparison with the corresponding sensing circles. Hence, there is a good chance that each sensor’s local coverage area is inside its MW-Voronoi region, which in turn means that the sensor does not need to move in order to increase its coverage area. On the other hand, when there are a large number of sensors in the target field, there is a good chance that each sensor covers its MW-Voronoi region, which implies that the termination condition will be satisfied in a short period of time. It is also to be noted that in the WVB algorithm the number of rounds required for the proper termination is larger than the other algorithms. The number of rounds in the Minmax-curve algorithm is relatively low, making it a good candidate for field coverage as far as the deployment time is concerned.

Another important means of assessing performance of sensor deployment algorithms is the energy consumption of the sensors. Sensors’ energy consumption highly depends on the traveling distance of sensors, and the number of times they stop before arriving at the destination (the latter is due to static friction). Thus, to compare the proposed methods in terms of energy consumption, the traveling distance and number of movements should be taken into consideration. Fig. 26 depicts the average moving distance for difference number of sensors. This figure shows that by increasing the number of sensors, the average moving distance of the sensors is decreased in all scenarios. This can be justified for each algorithm as follows. In the WVB algorithm, when the number of sensors increases, the distance between each sensor and its final position decreases.
This results in a decrease of average moving distance. In the other algorithms, on the other hand, when the number of sensors increases, the MW-Voronoi regions become smaller. As a result, the distance between each sensor and its destination point in the corresponding MW-Voronoi region decreases, which in turn leads to a decrease in the average moving distance. It can be concluded from Fig. 26 that the average moving distance of all eight algorithms are more or less the same when there are large number of sensors in the field. The number of movements versus the number of sensors is illustrated in Fig. 27. It can be observed from this figure that in general when the number of sensors is more than a certain level (whose value is different for each algorithm), the number of movements decreases. This is due to the fact that for large number of sensors the MW-Voronoi regions become smaller, which helps the sensors cover their MW-Voronoi regions. As a result, the coverage holes will be covered in a shorter period of time, decreasing the number of movements.

V. CONCLUSION

This paper presents efficient sensor deployment algorithms to improve coverage in mobile sensor networks. It is assumed that the sensing radii of different sensors are not the same. A multiplicatively weighted Voronoi diagram (MW-Voronoi diagram) is then employed to develop three distributed deployment algorithms accordingly. Under these algorithms, the sensors move iteratively to minimize coverage holes in the target field. Simulation results are pretested to compare the proposed approaches for different number of sensors in the target field.

REFERENCES

Fig. 26. The average distance each sensor travels for different number of sensors using the proposed algorithms.

Fig. 27. The number of movements required for different number of sensors using the proposed algorithms.


