Generalized Algebraic Connectivity for Asymmetric Networks

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Abstract—The problem of connectivity assessment of an asymmetric network represented by a weighted directed graph is investigated in this paper. The notion of generalized algebraic connectivity is introduced for this type of network as an extension of conventional algebraic connectivity measure for symmetric networks. This new notion represents the expected asymptotic convergence rate of a cooperative algorithm used to control the corresponding network. The proposed connectivity measure is then described in terms of the eigenvalues of the Laplacian matrix of the graph representing the network. The effectiveness of this measure in describing the connectivity of asymmetric networks is demonstrated through some intuitive and counter-intuitive examples. A power iteration algorithm is then developed to compute the proposed connectivity measure by properly transforming the Laplacian matrix of the network to a new matrix such that existing techniques can be used to find the eigenvalue representing network connectivity. The effectiveness of the proposed notion and also the efficiency of the developed algorithm are subsequently verified by simulations.

I. INTRODUCTION

Ad-hoc wireless networks are composed of a number of fixed or mobile nodes that are capable of exchanging data through wireless channels without the support of a pre-existing infrastructure [1], [2]. The convergence time of different cooperative algorithms used for objectives such as consensus, swarming, target localization and data diffusion is highly dependent on the degree of connectivity of the network [3], [4]. This is due to the fact that higher connectivity means that the network is capable of propagating information more effectively [5]. For the case of random networks where the communication channels are represented by random variables, the in-network information diffusion strictly depends upon the connectivity of the underlying expected communication graph [6].

The algebraic connectivity of an undirected network is defined in the literature as the smallest nonzero eigenvalue of the Laplacian matrix of the network graph [7]. A survey on algebraic connectivity measure for different types of networks and their computation techniques can be found in [8]. A decentralized orthogonal iteration algorithm is introduced in [9] for computing the eigenvectors corresponding to the k dominant eigenvalues of a symmetric weighted network graph. However, the procedure requires centralized initialization and is not scalable to larger network sizes. A distributed algorithm for the estimation and control of the algebraic connectivity of ad-hoc networks with a random topology is developed in [10], which is only applicable to symmetric networks. A generalization of Fiedler’s algebraic connectivity to directed graphs is introduced in [11], where several relationships between the algebraic connectivity and properties of the graph are investigated. Note that unlike symmetric networks, no standard definition exists for the rate of convergence to consensus in asymmetric networks; therefore, the notion of algebraic connectivity is not well developed for this class of networks. The notion of algebraic connectivity is extended to directed graphs in [12], where the magnitude of the second smallest eigenvalue of the Laplacian matrix is introduced as a measure of network connectivity. This notion, however, fails to capture important characteristics of the asymmetric network such as the convergence rate of cooperative algorithms used to control the network. Moreover, the decentralized power iteration approach proposed in [12] requires the solution of a set of nonlinear equations with relatively high computational complexity, which limits the applicability of the algorithm.

While there has been a growing interest in the application of asymmetric networks such as underwater acoustic sensor networks [13], [14], most of the papers cited in the previous paragraph are concerned with the connectivity of symmetric networks. As far as asymmetric networks are concerned, existing results fail to provide a tangible measure for the connectivity of the network. For example, it is noted in [3] that the addition of new edges in a directed graph representing an asymmetric network does not necessarily improve its connectivity as observed from the rate of convergence to consensus. However, this observation was not quantitatively characterized in the context of connectivity. Furthermore, the measure given in [12] is straightforward extension of the algebraic connectivity metric for symmetric networks, and does not reflect specific characteristics of the asymmetric networks in terms of the rate of convergence to consensus. The generalized algebraic connectivity is introduced in this work as a novel measure of connectivity of asymmetric networks represented by weighted directed graphs. The proposed connectivity measure captures the expected convergence speed of cooperative algorithms (e.g., consensus algorithms) used to control this type of networks. The effectiveness of the proposed measure over conventional algebraic connectivity measure in describing the connectivity of asymmetric networks is then demonstrated through an illustrative example. An example is also given to demonstrate the counter-intuitive relationship between the generalized algebraic connectivity
and the elements of the weight matrix of an asymmetric network. The generalized power iteration algorithm is then developed based on the Krylov subspace approximation method, Gram-Schmidt orthonormalization procedure, and a novel matrix transformation. The simulations confirm the efficacy of the results.

The remainder of the paper is organized as follows. Some background and definitions are given in Section II. The generalized algebraic connectivity of an asymmetric network is then introduced in Section III. Section IV presents the generalized power iteration algorithm to compute the proposed connectivity measure in a centralized fashion. The simulation results are subsequently presented in Section V, and the conclusions are summarized in Section VI.

II. PRELIMINARIES AND NOTATIONS

Throughout this work, the set of positive and nonnegative real numbers are denoted by \( \mathbb{R}_{>0} \) and \( \mathbb{R}_{\geq 0} \), respectively. Also \( \mathbb{N}_n \) is the finite set of natural numbers \( \{1, 2, \ldots, n\} \). The inner product of two arbitrary vectors \( v, w \in \mathbb{C}^n \) is denoted by \( \langle v, w \rangle \). Also, \( v^T \) and \( v^H \) represent the transpose and conjugate transpose of a vector \( v \in \mathbb{C}^n \), respectively. The \( n \times n \) identity matrix is denoted by \( \mathbf{I}_n \), and the all-one column vector of length \( n \) is represented by \( \mathbf{1}_n \). Also, \( \mathbf{1}_A : \mathcal{X} \to \{0, 1\} \) is the characteristic function over a set \( A \subseteq \mathcal{X} \), which is defined as

\[
\mathbf{1}_A(a) = \begin{cases} 
1, & \text{if } a \in A, \\
0, & \text{if } a \notin A. 
\end{cases}
\]

(1)

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{P}) \) represent a weighted directed graph (digraph) composed of \( n \) vertices with node set \( \mathcal{V} \), edge set \( \mathcal{E} \), and weight matrix \( \mathbf{P} = [p_{ij}] \) such that

\[
\mathcal{V} = \{1, 2, \ldots, n\},
\]

(2a)

\[
\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} | p_{ij} \neq 0\},
\]

(2b)

where \( p_{ij} \in \mathbb{R}_{\geq 0} \) for any ordered pair of distinct nodes \( (i, j) \in \mathcal{E} \). The node \( j \) is said to be a neighbor of node \( i \), denoted by \( \mathcal{N}_i \), if the directed edge pointing from \( j \) to \( i \) belongs to the edge set of \( \mathcal{G} \), i.e., \( (j, i) \in \mathcal{E} \). The Laplacian of the weighted digraph \( \mathcal{G} \) is an \( n \times n \) matrix \( \mathbf{L} = [l_{ij}] \) whose elements are given by

\[
l_{ij} = \begin{cases} 
-p_{ij}, & \text{if } (j, i) \in \mathcal{E}, \\
\sum_{k \neq i} p_{ik}, & \text{if } j = i, \\
0, & \text{otherwise}. 
\end{cases}
\]

(3)

Define \( \Psi(\mathbf{A}) \) as a set of triplets composed of the eigenvalues and the right and left eigenvectors of an asymmetric matrix \( \mathbf{A} \in \mathbb{R}^{n \times n} \) given by

\[
\Psi(\mathbf{A}) = \{ (\lambda_i(\mathbf{A}), v_i(\mathbf{A}), w_i(\mathbf{A})) | i \in \mathbb{N}_n \},
\]

(4)

where \( \lambda_i(\mathbf{A}) \in \mathbb{C} \), \( v_i(\mathbf{A}) \in \mathbb{C}^n \), and \( w_i(\mathbf{A}) \in \mathbb{C}^n \) denote the \( i \)-th eigenvalue of matrix \( \mathbf{A} \), and the right and left eigenvectors associated with it, respectively. This means that for any \( i \in \mathbb{N}_n \) one has

\[
\mathbf{A} v_i(\mathbf{A}) = \lambda_i(\mathbf{A}) v_i(\mathbf{A}),
\]

(5a)

\[
\mathbf{A}^T w_i(\mathbf{A}) = \lambda_i(\mathbf{A}) w_i(\mathbf{A}).
\]

(5b)

The triplets of \( \Psi(\mathbf{A}) \) are assumed to be indexed in increasing order in terms of the magnitude of the real parts of the eigenvalues, i.e.,

\[
\Re(\lambda_1(\mathbf{A})) \leq \Re(\lambda_2(\mathbf{A})) \leq \cdots \leq \Re(\lambda_n(\mathbf{A})).
\]

(6)

The spectrum of matrix \( \mathbf{A} \) is defined as \( \Lambda(\mathbf{A}) = \{ \lambda_i(\mathbf{A}) | i \in \mathbb{N}_n \} \).

III. GENERALIZED ALGEBRAIC CONNECTIVITY

The notion of generalized algebraic connectivity is introduced in this section as a new measure of connectivity for asymmetric networks. Connectivity has a significant impact on the diffusion of information throughout a network where each node shares information only with its neighbors. Nodes communicate more effectively in a more connected network, in general, and thus the information propagates faster throughout the network. For instance, when the nodes are supposed to agree upon a value of interest as a cooperative goal, the agreement can be reached with a higher rate of convergence in a network with a higher degree of connectivity. Therefore, the rate of convergence to consensus in a networked control system is highly dependent on its degree of connectivity.

Let the network be described by a weighted digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{P}) \) with node set \( \mathcal{V} \) and edge set \( \mathcal{E} \), where \( \mathbf{P} = [p_{ij}] \) is the weight matrix comprising the weights associated with the edges of the digraph. Let also \( q_t = (q_{ij})_{i \in \mathcal{N}_i} \) denote the \( d \)-dimensional state vector of any node \( i \in \mathcal{V} \), and consider the following dynamics for the \( i^\text{th} \) node

\[
\frac{d}{dt} q_i(t) = u_i(t),
\]

(7)

where \( u_i(t) \in \mathbb{R}^d \) is the control vector of node \( i \) at time \( t \). The following consensus control law can now be applied to the \( i^\text{th} \) node

\[
u_i(t) = -\sum_{j \in \mathcal{N}_i} p_{ij}(q_i(t) - q_j(t)).
\]

(8)

Note that \( p_{ij} \) is a nonnegative real number, and if it is nonzero, it means that there is an edge from node \( j \) to node \( i \) in \( \mathcal{G} \). Now, let \( \mathbf{q} = [q_1^T \ q_2^T \ \ldots \ q_n^T]^T \in \mathbb{R}^d \) denote the state vector of the entire network composed of \( n \) nodes, containing information of all nodes. The dynamics of the entire network driven by the consensus protocol (8) can then be represented as

\[
\frac{d}{dt} \mathbf{q}(t) = -(\mathbf{L} \otimes \mathbf{1}_d) \mathbf{q}(t),
\]

(9)

where \( \mathbf{L} \) denotes the Laplacian matrix of \( \mathcal{G} \) and \( \otimes \) denotes the Kronecker product. For simplicity of analysis, let the weighted digraph \( \mathcal{G} \) be strongly connected, which means that zero eigenvalue of \( \mathbf{L} \) has multiplicity one [15]. By definition, \( w_1(\mathbf{L}) \in \mathbb{R}^n \) is the left eigenvector of the Laplacian matrix \( \mathbf{L} \) associated with its zero eigenvalue. The final state vector that all nodes will converge to is denoted by \( \mathbf{q}_f = (\mathbf{q}_f)_j \in \mathcal{N}_i \), with

\[
\mathbf{q}_f = \frac{\langle w_1(\mathbf{L}), \mathbf{q}_i(0) \rangle}{\langle w_1(\mathbf{L}), \mathbf{1}_n \rangle}.
\]

(10)
for any $j \in \mathbb{N}_d$, where $q_{i,j} := (q_{i,j})_{i \in \mathbb{N}_n}$ is an $n$-dimensional real vector. To analyze the behavior of the network while converging to consensus, consider the following disagreement function

$$\vartheta_p(t) = \Gamma_p(q(t)), \quad (11)$$

where $\Gamma_p : \mathbb{R}^{nd} \rightarrow \mathbb{R}$ is defined as $\Gamma_p(q) = \|q - (1_n \otimes q_i)\|_p$ for any $p \in [1, \infty]$. The above disagreement function represents the $p$-norm of the difference between the state vector of the network and the final agreement value at time $t$. Furthermore, if the nodes take random vectors as their initial states, one can investigate the expected behavior of the disagreement function in order to measure the connectivity of the network. This point will be clarified by two illustrative examples.

**Example 1:** Consider two asymmetric networks represented by weighted digraphs $G_1$ and $G_2$ in Fig. 1. For each network, let the initial state be a random vector with a uniform distribution over the unit sphere in $\mathbb{R}^3$. Let also the nodes obey the dynamics (7), and apply the consensus control law (8) to every node. Fig. 2 depicts the evolution of function $E\vartheta(t)$ for the consensus algorithm running over networks 1 and 2. This figure indicates that

![Fig. 1. The weighted digraphs $G_1$ (left) and $G_2$ (right) in Example 1.](image1)

![Fig. 2. Time evolution of the expected disagreement function in Example 1.](image2)

the convergence to consensus in $G_2$ is expected to be faster than $G_1$, and thus, the weighted digraph $G_2$ is more connected than $G_1$.

**Remark 1:** It is worth noting that despite the faster expected convergence of $G_2$ to consensus in Example 1, the algebraic connectivity of $G_1$ is greater than that of $G_2$ [7]. This implies that existing algebraic connectivity measure fails to accurately reflects the connectivity of asymmetric networks, signifying the need for developing a new algebraic connectivity measure for digraphs.

**Example 2:** Consider the weighted digraphs $G_1$ and $G_2$ shown in Fig. 3, representing two asymmetric networks. According to Fig. 3, $G_1$ and $G_2$ have the same node set but $G_2$ has an additional edge pointing from node 2 to node 1 with weight 1. Intuitively, one would expect that $G_2$ has a higher degree of connectivity compared to $G_1$, and hence the expected disagreement function $E\vartheta(t)$ should converge to zero faster over the network represented by $G_2$. Let each node have a single-integrator dynamics given by (7), and apply the consensus control law (8) to every node. Fig. 4 depicts the time evolution of $E\vartheta(t)$ in the consensus problem over both networks. Surprisingly, the expected convergence rate

![Fig. 3. The weighted digraphs $G_1$ (left) and $G_2$ (right) in Example 2.](image3)

![Fig. 4. Time evolution of the expected disagreement function in Example 2.](image4)
guaranteed that $q$ converges to $1_n \otimes \tilde{q}$ as $t$ goes to infinity, i.e., $\lim_{t \to \infty} \delta_p(t) = 0$. For a more detailed analysis of this convergence, one can check the asymptotic exponential rate defined as $-\lim_{t \to \infty} t^{-1} \ln \delta_p(t)$. To this end, assume for simplicity and without loss of generality that the eigenvalues of the Laplacian matrix $L$ are distinct. Let the set of right eigenvectors and left eigenvectors of $L$ be chosen such that the vectors in each set are mutually orthonormal. Since $L$ is a diagonally dominant nonzero matrix, there exists a strictly increasing sequence of $m$ positive real scalars, denoted by $(r_k)_{k=1}^m$ for $m \in \mathbb{N}$, such that

$$\{r_k \mid k \in \mathbb{N}\} = \{\lambda_i(L) \mid \lambda_i(L) \in \Lambda(L), \lambda_i(L) \neq 0\}. \quad (12)$$

Note that the equality $e^{-Lt}v_i(L) = e^{-L(L)}t^r_i(L)$ holds at all times.

In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space, with $\Omega$, $\mathcal{F}$ and $\mathbb{P}$ denoting the sampling space, $\sigma$-algebra and probability measure, respectively. Let the initial state $q(0)$ be a random vector defined over $(\Omega, \mathcal{F})$, taking values in the unit sphere in $\mathbb{R}^{nd}$, denoted by $S^{nd-1}$. This means that there exists a random variable $X_{ij} : (\Omega, \mathcal{F}) \to \mathbb{C}$ for any $i \in \mathbb{N}$ and $j \in \mathbb{N}$, such that

$$q_{ij}(0) = \sum_{i \in \mathbb{N}} X_{ij} v_i(L). \quad (13)$$

It results from (9) that for any $j \in \mathbb{N}$,

$$\frac{d}{dt} q_{ij}(t) = -L q_{ij}(t). \quad (14)$$

It then follows that for any $t$

$$q_{ij}(t) = e^{-Lt} q_{ij}(0). \quad (15)$$

Also, one can easily verify that

$$\|q(t) - (1_n \otimes \tilde{q})\|_p = \sum_{j \in \mathbb{N}} \|q_{ij}(t) - \tilde{q}_{ij} 1_n\|_p, \quad (16)$$

for any $p \in [1, \infty]$. Therefore,

$$\delta^p(t) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} X_{ij} e^{-Lt} v_i(L) - \tilde{q}_{ij} 1_n, \quad (17)$$

From (10) and (13) and on noting that $\langle w_1(L), v_i(L) \rangle = 0$ for all $i \in \mathbb{N} \setminus \{1\}$, one arrives at

$$\tilde{q}_j = \left( \frac{\langle w_1(L), v_i(L) \rangle}{\langle w_1(L), 1_n \rangle} \right) = \sum_{i \in \mathbb{N}} X_{ij} \frac{\langle w_1(L), v_i(L) \rangle}{\langle w_1(L), 1_n \rangle} = X_{ij}, \quad (18)$$

(note that $v_1(L) = 1_n$ because zero is an eigenvalue of $L$ with multiplicity one as noted earlier). Since $\lambda_1(L) = 0$, it follows from (17) and (18) that

$$\delta^p(t) = \sum_{j \in \mathbb{N}} \sum_{i=2}^n X_{ij} e^{-Lt} v_i(L)\|_p. \quad (19)$$

For any $k \in \mathbb{N}$, define the index set $I_k$ as

$$I_k = \{i \in \mathbb{N} \mid \mathbb{R}(\lambda_i(L)) = r_k\}, \quad (20)$$

which contains the index of any eigenvalue of $L$ whose real part is equal to $r_k$. Let the event $A_k \in \mathcal{F}$ be defined as

$$A_k = \bigcup_{i \in I_k, j \in \mathbb{N}} \{X_{ij} \neq 0\}, \quad (21)$$

for any $k \in \mathbb{N}$. An important property of $A_k$ is that for at least one $j \in \mathbb{N}$, there exists a nonzero vector $v_i(L)$ such that $i \in I_k$ and the inner product of $q_{ij}(0)$ and $v_i(L)$ is nonzero. Now, define the events $\{B_k\}_{k=1}^m \subset \mathcal{F}$ such that $B_1 = A_1$ and

$$B_k = A_k - \bigcup_{l=1}^{k-1} A_l, \quad (22)$$

for any $k \in \mathbb{N} \setminus \{1\}$. Since the sets $\{B_k\}_{k=1}^m$ are mutually disjoint and also $\Omega = \bigcup_{k=1}^m B_k$, they partition the sample space $\Omega$ (which means that their intersection is empty and their union is the whole sample space). Let the random variable $r$, referred to as the random rate, be defined as $r := \sum_{k \in \mathbb{N}} r_k 1_{B_k}$. Define also

$$S_{jk}(t) = \sum_{k' \geq k \in I_k} \sum_{i=1}^n X_{ij} e^{-L(L)\tau_r} v_i(L), \quad (23)$$

for any $j \in \mathbb{N}$ and $k \in \mathbb{N}$. It can then be verified that

$$\langle e^{rt} \delta_p(t) \rangle = \sum_{j \in \mathbb{N}} \sum_{k' \geq k \in I_k} \sum_{i=2}^n X_{ij} e^{-Lt} v_i(L)\|_p^p = \sum_{j \in \mathbb{N}} \|S_{jk}(t) 1_{B_k}\|_p^p \quad (24)$$

Note that the real part of $-\lambda_1(L) + r_k$ is nonpositive for any $k, k' \in \mathbb{N}$, $k \leq k'$. For any $i \in I_k$. It then follows from the triangle inequality that

$$\|S_{jk}(t)\|_p \leq \sum_{k' \geq k \in I_k} \sum_{i=1}^n \|X_{ij}\|_p\|v_i(L)\|_p. \quad (25)$$

One can subsequently conclude that for any $t$

$$\langle e^{rt} \delta_p(t) \rangle \leq \sum_{j \in \mathbb{N}} \big( \sum_{k' \geq k \in I_k} \sum_{i=1}^n \|X_{ij}\|_p\|v_i(L)\|_p \big) = \sum_{j \in \mathbb{N}} \|S_{jk}(t) 1_{B_k}\|_p^p \quad (26)$$

Lemma 1: Let $n$ be a positive integer and $l \in \mathbb{N}$. Let also $\{v_i\}_{i \in \mathbb{N}}$ denote a set of linearly independent unit vectors in $\mathbb{C}^n$, and $\{\gamma_i\}_{i \in \mathbb{N}}$ be a set of complex scalars with nonpositive real parts. Then, for any set of complex scalars $\{c_i\}_{i \in \mathbb{N}}$, the following relation holds

$$\liminf_{t \to \infty} \| \sum_{i \in \mathbb{N}} c_i e^{\gamma_i t} v_i \|_p = 0, \quad (27)$$

if and only if $c_i = 0$ for any $i \in \mathbb{N}$ such that $\Re(\gamma_i) = 0$.

Proof: Let the index set $N_l$ be partitioned into two disjoint sets $N_l^*$ and $N_l \setminus N_l^*$ for some $l^* \leq l$ based on the real parts of the elements of $\{\gamma_i\}_{i \in \mathbb{N}}$ such that $\Re(\gamma_i) = 0$ for all $i \in N_l^*$ and $\Re(\gamma_i) < 0$ for all $i \in \mathbb{N} \setminus N_l^*$. One can represent the term $\sum_{i \in \mathbb{N}} c_i e^{\gamma_i t} v_i$ in a matrix form, i.e.,

$$V_l = [v_1 v_2 \cdots v_l], \quad (28a)$$

$$C_l(t) = [c_1 e^{\gamma_1 t} c_2 e^{\gamma_2 t} \cdots c_l e^{\gamma_l t}], \quad (28b)$$

where $V_l \in \mathbb{C}^{n \times l}$ and $C_l(t) \in \mathbb{C}^{l \times 1}$ denote the matrix of unit vectors and the vector of coefficients, respectively. Using the triangle inequality, the term $\|V_l C_l(t)\|_p$ can be lower-bounded and upper-bounded in a matrix form as follows

$$\|V_l C_l(t)\|_p = \|V_l (I_l - C_{l-I_l}(t))\|_p \leq \|V_l C_l(t)\|_p, \quad (29a)$$

$$\|V_l C_l(t)\|_p \leq \|V_l C_l(t)\|_p + \|V_l (I_l - C_{l-I_l}(t))\|_p. \quad (29b)$$
Now consider the behavior of two inequalities given in (29) by taking \( \lim \inf \) as \( t \to \infty \). Since the \( i^{th} \) element of \( C_{t}^{i} \) is given by \( c_{i}e^{t\gamma_{i}} \) with \( \Re(\gamma_{i}) < 0 \) for any \( i \in \mathbb{N}_{r} \) and any \( t \), it is straightforward to conclude that \( \lim \inf_{t \to \infty} \left| V_{t}^{i} \cdot C_{t}^{i} \right| = 0 \). Using this fact and considering (29), one arrives at

\[
\lim \inf_{t \to \infty} \left\| V_{t}^{i} \cdot C_{t}^{i} \right\| \leq \lim \inf_{t \to \infty} \left\| V_{t}^{i} \right\|, \tag{30a}
\]

\[
\lim \inf_{t \to \infty} \left\| V_{t}^{i} \right\| \leq \lim \inf_{t \to \infty} \left\| V_{t}^{i} \cdot C_{t}^{i} \right\|, \tag{30b}
\]

The objective is to demonstrate that having \( c_{i} = 0 \) for all \( i \in \mathbb{N}_{r} \), is a necessary and sufficient condition for validity of equation (27). To prove the necessity of this condition, assume that \( c_{i} = 0 \) holds for all \( i \in \mathbb{N}_{r} \). This implies that \( C_{t}^{i} = 0_{r \times 1} \) for any \( t \), which yields \( \lim_{t \to \infty} \left\| V_{t}^{i} \cdot C_{t}^{i} \right\| = 0 \). It then follows from (30) that

\[
0 \leq \lim \inf \left\| V_{t}^{i} \right\| \
\]

which results in \( \lim_{t \to \infty} \left\| V_{t}^{i} \right\| = 0 \) and completes the proof. To prove the sufficiency, assume that one has

\[
\lim_{t \to \infty} \left\| V_{t}^{i} \right\| = 0.
\]

It then follows from (30) that

\[
\lim_{t \to \infty} \left\| V_{t}^{i} \cdot C_{t}^{i} \right\| = 0.
\]

In this case, \( \left\| C_{t}^{i} \right\| \) can be rewritten as

\[
\left\| C_{t}^{i} \right\| = \sum_{i=1}^{l^*} \left| c_{i} e^{t\gamma_{i}} \right| = \sum_{i=1}^{l^*} \left| c_{i} \right| \cdot \left( \sum_{i=1}^{l^*} \left| c_{i} \right| \right)^{T}.
\]

Now, let \( S \) denote a set of complex vectors in \( C^{r} \) such that \( S = \left\{ z \in C^{r} \mid ||z|| = \sum_{i=1}^{l^*} \left| c_{i} \right| \cdot \sum_{i=1}^{l^*} \left| c_{i} \right|^{T} \right\} \). Since \( S \) is bounded and closed, it is compact as well. Consider \( V_{t}^{i} : C^{r} \to C^{n} \) as a linear continuous map. Due to the continuity of \( V_{t}^{i} \), the set \( S \) is mapped into a compact set under \( V_{t}^{i} \). Since \( \lim_{t \to \infty} \left\| V_{t}^{i} \cdot C_{t}^{i} \right\| = 0 \), there exists a time sequence \( \{ t_{i} \}_{i=1}^{l^*} \) such that \( \left\| V_{t}^{i} \cdot C_{t}^{i} \right\| = 0 \) as \( t_{i} \to \infty \). Define \( \{ w_{i} \}_{i=1}^{l^*} \) as a sequence of vectors in \( C^{n} \) such that \( w_{i} = V_{t}^{i} \cdot C_{t}^{i} \) as \( t_{i} \to \infty \). It then follows that \( w_{i} \to 0_{l^* \times 1} \) as \( i \to \infty \). Due to the closedness of the mapping of \( S \) under \( V_{t}^{i} \), it can be concluded that there exists a vector like \( z^{*} \in C^{r} \) belonging to \( S \) such that \( V_{t}^{i} \cdot z^{*} = 0 \). Since the matrix \( V_{t}^{i} \) is composed of \( l^* \) linearly independent unit vectors which is full rank, one concludes that \( z^{*} = 0_{l^* \times 1} \), or \( ||z^{*}|| = 0 \). According to definition of \( S \) and since \( z^{*} \in S \), one has \( \sum_{i=1}^{l^*} |c_{i}| = 0 \), which results in \( c_{i} = 0 \) for all \( i \in \mathbb{N}_{r} \). This concludes the sufficiency of the lemma.

Since \( ||q(0)|| = 0 \) for any initial state, there exists \( i \in \mathbb{N}_{a} \) and \( j \in \mathbb{N}_{d} \) for which \( X_{ij} \neq 0 \). Let \( k \in \mathbb{N}_{b} \) be such that \( i \in I_{k} \). It then follows from (23) and Lemma 1 that

\[
\lim_{t \to \infty} \frac{\ln \delta_{p}(t)}{t} = r.
\]

If the initial state \( q(0) \) has a continuous probability distribution, one can show that \( \Pr(B_{1} = 1) = 1 \), which yields

\[
\Pr(- \lim_{t \to \infty} \frac{\ln \delta_{p}(t)}{t} = r_{1}) = 1. \tag{36}
\]

Following a similar argument, one can show that the same results hold for \( p = \infty \). The next theorem follows from the previous discussion.

**Theorem 1:** Let \( G \) be a strongly connected weighted digraph with Laplacian matrix \( L \) whose eigenvalues are distinct, and \( p \in [1, \infty] \) denote a real scalar. Let also \( q \) denote the state of a dynamical system described by (9), whose initial state \( q(0) \) is a random vector with a continuous probability distribution over \( \mathbb{R}^{n} \). Consider the disagreement function \( \delta_{p} \) given in (11), and define \( \lambda(L) = \min_{\lambda_{i}(L) \neq 0, \lambda_{i}(L) \in \Lambda(L)} \Re(\lambda_{i}(L)) \). Then,

(i) the asymptotic exponential rate of convergence to consensus is at least \( \lambda(L) \), i.e.,

\[
- \lim_{t \to \infty} t^{-1} \ln \delta_{p}(t) \geq \lambda(L);
\]

(ii) the asymptotic exponential rate of convergence to consensus is almost surely equal to \( \lambda(L) \), i.e.,

\[
\mathbb{P}[- \lim_{t \to \infty} t^{-1} \ln \delta_{p}(t) = \lambda(L)] = 1;
\]

(iii) the expectation of asymptotic exponential rate of convergence to consensus equals \( \lambda(L) \), i.e.,

\[
\mathbb{E}[- \lim_{t \to \infty} t^{-1} \ln \delta_{p}(t)] = \lambda(L).
\]

**Proof:** The proof of this theorem is a direct consequence of the above discussion.

**Theorem 1** shows that the convergence rate to consensus is well described by \( \lambda(L) \) as defined in the statement of the theorem. This motivates the introduction of a new connectivity measure for weighted digraphs.

**Definition 1:** Given a weighted digraph \( G \) with Laplacian matrix \( L \), the **generalized algebraic connectivity** of \( G \), denoted by \( \lambda(G) \), is defined as

\[
\lambda(G) = \min_{\lambda_{i}(L) \neq 0, \lambda_{i}(L) \in \Lambda(L)} \Re(\lambda_{i}(L)). \tag{37}
\]

The generalized algebraic connectivity (GAC) reflects the asymptotic convergence rate of cooperative consensus-based algorithms in an asymmetric network represented by a weighted digraph. It can be verified that in Example 1 \( \lambda(L_{1}) < \lambda(L_{2}) \) and in Example 2 \( \lambda(L_{1}) > \lambda(L_{2}) \), which are in accordance with the corresponding convergence results. In the next section, an algorithm is presented to compute the generalized algebraic connectivity.

**IV. COMPUTATION OF THE GENERALIZED ALGEBRAIC CONNECTIVITY**

An algorithm is presented in this section to compute the GAC of a weighted digraph. The algorithm is an extension of the well-known power iteration method which has been used extensively in the literature to compute the algebraic connectivity of undirected networks in both centralized and distributed fashions [10], [12]. The power iteration algorithm computes the dominant eigenvalue of a matrix (i.e., an eigenvalue with maximum magnitude) as well as the eigenvector associated with it, under certain conditions [16]. Some of the challenges in using the power iteration method to compute
the GAC are as follows:
(i) the power iteration method computes the eigenvector associated with the eigenvalue with maximum (not minimum) magnitude (not real part), and
(ii) the convergence of the procedure is not guaranteed when the eigenvalues associated with the GAC of a network are complex conjugate. In order to address the above challenges, some important results are developed in the sequel.

Lemma 2: Let $L$ denote the Laplacian matrix of a weighted digraph with $n$ nodes, and define $L' = I_n - \epsilon L$, where $\epsilon$ is a small positive constant. It then follows from (41) that
\begin{align*}
\lambda_i(L) = \frac{1}{\epsilon}(1 - \lambda_{n-i+1}(L')),
\end{align*}
\begin{align*}
v_i(L) = v_{n-i+1}(L'),
\end{align*}
\begin{align*}
w_i(L) = w_{n-i+1}(L'),
\end{align*}
for every $i \in \mathbb{N}_n$.

Proof: From the definition of the right and left eigenvectors, one has
\begin{align*}
LV(L) &= V(L) \Phi(L), \tag{39a}
\end{align*}
\begin{align*}
L^T W(L) &= W(L) \Phi(L), \tag{39b}
\end{align*}
where $V(L) = [v_1(L) v_2(L) \cdots v_n(L)]$, $W(L) = [w_1(L) w_2(L) \cdots w_n(L)]$, and $\Phi(L) = \text{diag} (\lambda_1(L), \ldots, \lambda_n(L))$, such that $\Re(\lambda_1(L)) \leq \Re(\lambda_2(L)) \leq \cdots \leq \Re(\lambda_n(L))$. Multiplying (39a) by $-\epsilon$ and adding $V(L)$ to both sides yields
\begin{align*}
V(L) - \epsilon L V(L) &= V(L) - \epsilon \Phi(L) V(L), \tag{40}
\end{align*}
or
\begin{align*}
(I_n - \epsilon L) V(L) &= V(L) (I_n - \epsilon \Phi)(L). \tag{41}
\end{align*}
Define $L' = I_n - \epsilon L$ with the diagonal eigenvalue matrix $\Phi(L') = I_n - \epsilon \Phi(L)$, and let $V(L')$ and $W(L')$ denote the matrices of right and left eigenvectors of $L'$, respectively. It then follows from (41) that
\begin{align*}
L' V(L') &= V(L') \Phi(L'). \tag{42}
\end{align*}
Sorting the eigenvalues of $L'$ in increasing order in terms of the magnitude of their real parts leads to a set of triplets $\Psi(L')$ in reverse order as compared to $\Psi(L)$. By comparing equations (41) and (42), it can be concluded that
\begin{align*}
\lambda_{n-i+1}(L') = 1 - \epsilon \lambda_i(L),
\end{align*}
or
\begin{align*}
\lambda_i(L) = \frac{1}{\epsilon}(1 - \lambda_{n-i+1}(L')), \tag{43}
\end{align*}
for any $i \in \mathbb{N}_n$. Moreover, it is implied from (41) and (42) that
\begin{align*}
v_i(L) = v_{n-i+1}(L'),
\end{align*}
\begin{align*}
w_i(L) = w_{n-i+1}(L'),
\end{align*}
for any $i \in \mathbb{N}_n$, which means that the set of right eigenvectors of $L$ and $L'$ are the same but are sorted in the opposite order in $V(L)$ and $V(L')$. Using a similar argument, it can be shown that
\begin{align*}
w_i(L) = w_{n-i+1}(L'),
\end{align*}
for any $i \in \mathbb{N}_n$. This completes the proof.

Remark 2: The positive constant $\epsilon$ used in the definition of $L' = I_n - \epsilon L$ in Lemma 2 should be chosen sufficiently small to ensure that $L'$ is a nonnegative matrix, i.e., $l'_{ij} \geq 0$ for any $i, j \in \mathcal{V}$. For instance, one can choose it as follows
\begin{align*}
\epsilon \leq \left( \max_{j \in \mathcal{V}} \sum_{i \in \mathcal{V}_j} p_{ij} \right)^{-1}. \tag{47}
\end{align*}
Let the inequality $0 \leq p_{ij} \leq p_{\max}$ hold for every element of the weight matrix $P = [p_{ij}]$ and for a positive constant $p_{\max}$. This leads to the following relation
\begin{align*}
\max_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}_i} p_{ij} = (n - 1)p_{\max}. \tag{48}
\end{align*}
It then follows from (47) and (48) that by choosing $\epsilon = (np_{\max})^{-1}$, it is guaranteed that $L'$ is a nonnegative matrix.

It is known that if $\lambda$ is an eigenvalue of matrix $A \in \mathbb{R}^{n \times n}$, then $e^{\lambda}$ is an eigenvalue of $e^{A}$. Furthermore, $v$ and $w$ are respectively the right and left eigenvectors of $A$ associated with $\lambda$, they are also the right and left eigenvectors of $e^{A}$ associated with $e^{\lambda}$ (this can be easily shown using the definition of eigenvalues and eigenvectors and the Taylor expansion of $e^{A}$). The following lemma is borrowed from [12], and will be used in the proof of the main result.

Lemma 3: Let $v_i(A)$ and $w_i(A)$ be the right and left eigenvectors of matrix $A \in \mathbb{R}^{n \times n}$ associated with the eigenvalue $\lambda_i(A)$. Define the deflated matrices $A_{L_i}$ and $A_{R_i}$ by the following affine transformations
\begin{align*}
A_{L_i} = A - \lambda_i(A) v_i(A) w_i^T(A), \tag{49a}
A_{R_i} = A - \lambda_i(A) v_i(A) w_i^T(A). \tag{49b}
\end{align*}
It then follows that
\begin{align*}
\Lambda(A_{L_i}) = \Lambda(A_{R_i}) = (\Lambda(A) \setminus \{\lambda_i(A)\}) \cup \{0\}, \tag{50a}
v_j(A_{L_i}) = v_j(A), \tag{50b}
w_j(A_{R_i}) = w_j(A), \tag{50c}
\end{align*}
for any $j \in \mathbb{N}_n \setminus \{i\}$.

Theorem 2: Let $L$ be the Laplacian matrix of a strongly connected weighted digraph $\mathcal{G}$ composed of $n$ nodes, and define the modified Laplacian matrix of the digraph as
\begin{align*}
\check{L} = e^{I_n - \epsilon L} - \exp(1)w_1 w_1^T. \tag{51}
\end{align*}
Then the following relations hold
\begin{align*}
\check{\lambda}(L) &= \frac{1}{\epsilon}(1 - \ln(\max_{i \in \mathcal{V}} |\lambda_i(L)|)), \tag{52a}
\check{v}(L) &= v_n(L), \tag{52b}
\end{align*}
where $\check{v}(L)$ denotes the right eigenvector of $L$ whose associated eigenvalue determines the generalized algebraic connectivity of $\mathcal{G}$.

Proof: According to Lemma 2, the spectrum of matrix $L' = I_n - \epsilon L$ is given by $\Lambda(L') = \{\lambda_i(L')\}$, $\lambda_i(L') = \frac{1}{\epsilon}(1 - \lambda_{n-i+1}(L'))$, $i \in \mathbb{N}_n$, and by definition its elements are sorted in the following order
\begin{align*}
\Re(\lambda_1(L')) \leq \Re(\lambda_2(L')) \leq \cdots \leq \Re(\lambda_n(L')). \tag{53}
\end{align*}
It follows from Lemma 2 and the definition of the generalized algebraic connectivity that \( \lambda(L) \) can be reformulated as

\[
\hat{\lambda}(L) = \frac{1}{\epsilon} (1 - \lambda_i(L') \neq 1, \lambda_i(L')) \text{Re}(\lambda_i(L')).
\]  

(54)

A transformation is now required to relate the real part of the eigenvalues of a matrix to the magnitude of the eigenvalues of another matrix. Using (53), one can write

\[
e^{\text{Re}(\lambda_i(L'))} \leq e^{\text{Re}(\lambda_i(L'))} \leq \ldots \leq e^{\text{Re}(\lambda_i(L'))},
\]

(55)

Let the \( i \)th element of \( \Lambda(L') \) be decomposed into its real and imaginary parts as \( |\lambda_i(L')| = | \text{Re}(\lambda_i(L')) + j \text{Im}(\lambda_i(L')) | \). Since \( |\lambda_i(L')| = 1 \) for any \( i \in \mathbb{N}_n \), the inequality (55) can be rewritten as

\[
e^{\text{Re}(\lambda_i(L')) + j \text{Im}(\lambda_i(L'))} \leq e^{\text{Re}(\lambda_i(L')) + j \text{Im}(\lambda_i(L'))} \leq \ldots \leq e^{\text{Re}(\lambda_i(L')) + j \text{Im}(\lambda_i(L'))},
\]

(56)

or, equivalently

\[
e^{\lambda_i(L')} = e^{\lambda_i(L')} = \ldots \leq e^{\lambda_i(L')}.
\]

(57)

Note that the spectrum of matrix \( e^{L'} \) is given by \( \Lambda(e^{L'}) = \{ \lambda_i(e^{L'}) \mid i \in \mathbb{N}_n \} \). Hence, it follows from (57) that the eigenvalues of \( e^{L'} \) are indexed in increasing order in terms of their magnitude. Since the weighted digraph \( \mathcal{G} \) is strongly connected, \( \lambda_n(L') = 1 \) and \( \lambda_e(e^{L'}) = \text{exp}(1) \). Thus,

\[
\lambda_i(L') \neq 1, \lambda_i(L') \in \Lambda(L')
\]

\[
\text{max} \lambda_i(L') \in \Lambda(L') \text{Re}(\lambda_i(L')) = \ln \left( \text{max} \lambda_i(L') \in \Lambda(L') | \lambda_i(e^{L'}) \right).
\]

(58)

The modified Laplacian matrix \( \tilde{L} \) is then obtained by subtracting the deflation term \( \text{exp}(1)w_1 w_1^T \) from \( e^{L'} \), which zeros the eigenvalue \( \text{exp}(1) \) of the matrix \( e^{L'} \) in its spectrum according to Lemma 3. As a result, (58) can be written as follows

\[
\max \lambda_i(L') \neq 1, \lambda_i(L') \in \Lambda(L') \text{Re}(\lambda_i(L')) = \ln \left( \text{max} \lambda_i(L') \in \Lambda(L') | \lambda_i(e^{L'}) \right).
\]

(59)

Equations (52a) now follow from (54) and (59). From Lemmas 2 and 3, and on noting that the left eigenvector \( w_1 \) associated with the zero eigenvalue of \( L \) is used in the deflation term in the modified Laplacian matrix \( \tilde{L} \), it follows that the right eigenvector \( \tilde{v}(L) \) whose associated eigenvalue determines the generalized algebraic connectivity of \( L \) is the same as the right eigenvector associated with an eigenvalue of \( L \) with the largest magnitude. In other words, \( \tilde{v}(L) = v_n(L) \), and this completes the proof.

The generalized power iteration (GPI) algorithm is developed here as an extension of the power iteration method to compute the GAC measure of asymmetric networks. To address the first challenge noted in the beginning of this section, the GPI algorithm is applied to the modified Laplacian matrix \( \tilde{L} \) which is obtained by transforming \( L \) to another matrix of the same size as follows

\[
\tilde{L} = e^{L} - \epsilon L - \text{exp}(1)w_1 w_1^T,
\]

(60)

where \( \epsilon \) is a sufficiently small positive constant as discussed in Remark 2. It then follows from Theorem 2 that an eigenvalue of \( L \) with maximum magnitude is, in fact, the transformed version of an eigenvalue of \( L \) with the second smallest real part. Now, to address the second challenge, a variant of the Krylov subspace method is used to develop the GPI algorithm in order to compute a relatively good approximation of the GAC, no matter if the corresponding eigenvalue is real or complex. The Krylov subspace procedure is briefly introduced next, followed by the proposed generalized power iteration algorithm.

A. Krylov Subspace Method [16]

The \( m \)-dimensional Krylov subspace of \( \mathbb{C}^n \) w.r.t. the vector \( x_0 \in \mathbb{C}^n \) and matrix \( L \in \mathbb{R}^{n \times n} \), denoted by \( K_m(x_0, L) \) for \( m \ll n \), is defined as

\[
K_m(x_0, L) = \text{span}\{x_0, Lx_0, \ldots, L^{m-1}x_0\}.
\]

(61)

Let \( Q_m = [\hat{x}_1 \ldots \hat{x}_m] \) denote a \( n \times m \) matrix whose \( m \) columns constitute an orthonormal basis of \( K_m(x_0, L) \) such that \( Q_m^H \hat{Q}_m = I_m \). Let also \( R_m \) represent the orthogonal projection of \( L \) into the \( m \)-dimensional Krylov subspace such that

\[
R_m = Q_m^H L Q_m.
\]

(62)

A Krylov subspace method approximates \( s \) eigenpairs of \( L \) by those of the reduced matrix \( R_m \) of order \( m \), where \( s \ll n \). Then, the eigenpair \( (\lambda, Q_m \hat{v}) \) provides an approximation of the eigenpair \( (\lambda, v) \) of \( L \), such that

\[
R_m \hat{v} = \hat{\lambda} \hat{v}.
\]

(63)

To handle the case where the GAC measure of a network is associated with a pair of complex conjugate eigenvalues of \( L \), it is required to compute the eigenpairs of the first two dominant eigenvalues of \( L \). To this end, one can consider \( s = 2 \) and \( m \geq 2 \) in the GPI algorithm. To generate the orthonormal columns of \( Q_m \) using the Gram-Schmidt process, consider an arbitrary initial vector \( x_0 \) and define \( \hat{x}_1 = \frac{x_0}{\|x_0\|} \). Then, \( \hat{x}_{j+1} \) is obtained as

\[
\hat{x}_{j+1} = \frac{\tilde{L} \hat{x}_j - \sum_{i=1}^{j} (\tilde{L} \hat{x}_j, \hat{x}_i) \hat{x}_i}{\|\tilde{L} \hat{x}_j - \sum_{i=1}^{j} (\tilde{L} \hat{x}_j, \hat{x}_i) \hat{x}_i\|}, \quad j \in \mathbb{N}_{m-1}.
\]

(64)

The accuracy of the approximate eigenpair \( (\hat{\lambda}, Q_m \hat{v}) \) for the original matrix \( L \) is assessed by the residual norm \( \rho \), which is defined as

\[
\rho = \| (\tilde{L} - \hat{\lambda} I_n) Q_m \hat{v} \|.
\]

(65)

The Krylov subspace method is used in the GPI algorithm to iteratively solve the maximization problem described in (52a). A procedure for centralized implementation of the GPI algorithm is elaborated in Algorithm 1.

Theorem 3: Let \( \hat{\lambda}(L') \) denote the GAC measure of a strongly connected weighted digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{P}) \) composed of \( n \) nodes with Laplacian matrix \( L \) obtained by applying the GPI algorithm. Denote \( \hat{v} \) as the right eigenvector associated with an eigenvalue of \( L \) representing the GAC of \( \mathcal{G} \), which is obtained from the GPI algorithm. Assume that
Algorithm 1 Centralized Generalized Power Iteration
1. Choose \( s = 2, m \in \mathbb{N}_{+} \), and let \( v(0) = \text{rand}(n, 1) \) be the initial right eigenvector associated with the eigenvalue representing the GAC of \( \mathcal{G} \). Let also \( \rho_0 \) be a sufficiently small value used in the termination criterion.

2. \( k = 1; x_0 = v(0) \).

3. while \( \rho(k) > \rho_0 \) do

4. \( \tilde{L} = \sum_{i=0}^{l} \frac{1}{n!}(I_n - cL)^i - \exp(1) w_1(L) w_1^T(L) \).

5. Construct the matrix \( Q_m(k) \) based on the Gram-Schmidt method using (64) to represent the Krylov subspace w.r.t. vector \( x_0 \) and matrix \( \tilde{L} \).

6. Construct the matrix \( R_m(k) = Q_m^H(k) \tilde{L} Q_m(k) \), which is the projection of \( \tilde{L} \) into the subspace spanned by the columns of \( Q_m(k) \).

7. Compute \( \hat{\Lambda}(k) = \text{diag}(\hat{\lambda}_1(k), \ldots, \hat{\lambda}_m(k)) \) and \( \hat{V}(k) = [\hat{v}_1(k) \cdots \hat{v}_m(k)] \) such that \( R_m(k) \hat{V}(k) = \hat{V}(k) \hat{\Lambda}(k) \).

8. \( \hat{\lambda}(k) = \frac{1}{\epsilon}(1 - \ln((\hat{\lambda}_i^*(k)))) \) and \( \hat{v}(k) = Q_m(k) \hat{v}_i^*(k) \) where \( i^* = \text{argmax}_{i \in \mathbb{N}_m} |\hat{\lambda}_i(k)| \).

9. \( \rho(k) = ||\tilde{L}\hat{v}(k) - \hat{\lambda}_i^*(k) \hat{v}(k)|| \).

10. \( x_0 = \hat{v}(k); k = k + 1 \).

end while

return \( \hat{\lambda}(k), \hat{v}(k) \).

The GAC measure of \( L \) obtained from Algorithm 1 is also denoted by \( \hat{\lambda}(L') \). Define \( \hat{\lambda}(L) \) and \( \hat{\lambda}(L') \) as eigenvalues with largest magnitude associated with matrices \( L \) and \( L' \), respectively. Let also \( \hat{\lambda}(L'') \) denote the output of the Krylov subspace approximation method obtained from Algorithm 1, which represents an eigenvalue of \( L' \) with largest magnitude. According to the definition of the residual norm in (65) and definition of \( \rho_0 \), one has

\[
\left| \hat{\lambda}(L') - \hat{\lambda}(L'') \right| \leq \frac{\rho_0}{||\hat{v}||}. \tag{71}
\]

where \( \hat{v} \) denotes the right eigenvector associated with an eigenvalue of \( L' \) with largest magnitude obtained from Algorithm 1. Since the matrix exponential term in definition of \( L' \) and \( L'' \) is approximated by the first \( l \) terms of its Taylor expansion, (71) can be rewritten as follows

\[
\left| \sum_{i=1}^{l} \frac{1}{i!} [(1 - \epsilon \hat{\lambda}(L'))^i - (1 - \epsilon \hat{\lambda}(L''))^i] \right| \leq \frac{\rho_0}{||\hat{v}||}. \tag{72}
\]

Note that the following equality holds for any \( i \in \mathbb{N}_l \) and any \( a, b \in \mathbb{R}_{>0} \)

\[
a^i - b^i = (a - b) \sum_{j=1}^{i} a^{i-j} b^{j-1}. \tag{73}
\]

Using (73) and on noting that \( 1 - \epsilon \hat{\lambda}(L') \) and \( 1 - \epsilon \hat{\lambda}(L'') \) are nonnegative (due to the chosen sufficiently small positive constant \( \epsilon \) specified in Remark 2), (72) is simplified as follows

\[
\epsilon \left| \hat{\lambda}(L'') - \hat{\lambda}(L') \right| \sum_{i=1}^{l} \frac{1}{i!} \sum_{j=1}^{i} (1 - \epsilon \hat{\lambda}(L'))^{i-j} \times (1 - \epsilon \hat{\lambda}(L''))^{j-1} \leq \frac{\rho_0}{||\hat{v}||}. \tag{74}
\]

It yields from (74) that

\[
\left| \hat{\lambda}(L'') - \hat{\lambda}(L') \right| \leq \frac{\rho_0}{||\hat{v}||} \left( \epsilon \sum_{i=1}^{l} \frac{1}{i!} \sum_{j=1}^{i} (1 - \epsilon \hat{\lambda}(L'))^{i-j} \times (1 - \epsilon \hat{\lambda}(L''))^{j-1} \right)^{-1}. \tag{75}
\]

Define \( E_1 \) as an upper bound on approximation error of the GPI algorithm resulted from the inaccuracy of the Krylov subspace method used in Algorithm 1. According to (75) and on noting that \( np_{\text{max}} \) represents an upper bound on magnitude of the eigenvalues of \( L \), \( E_1 \) can be obtained as follows

\[
E_1 = \frac{\rho_0}{||\hat{v}||} \left( \epsilon \sum_{i=1}^{l} \frac{1}{i!} \sum_{j=1}^{i} (1 - np_{\text{max}})^{i-j} \times (1 - np_{\text{max}})^{j-1} \right)^{-1}, \tag{76}
\]

where

\[
\left| \hat{\lambda}(L'') - \hat{\lambda}(L') \right| \leq E_1. \tag{77}
\]

Now, one should investigate the effect of considering a finite number of terms in the Taylor expansion of the exponential
of Laplacian matrix $L$. It is straightforward to see that the following equality holds for any $\lambda_i \in \Lambda(L)$, $i \in \mathbb{N}_n$

$$\lambda_i(L) = \frac{1}{\epsilon}(1 - \ln(e^{1-\epsilon \lambda_i})). \quad (78)$$

Let $\lambda_i(L')$ denote the $i^{th}$ eigenvalue of $L'$ where the term $e^{L'}$ in its definition is approximated by the first $l$ terms of its Taylor expansion. Then one has

$$\lambda_i(L') = \frac{1}{\epsilon}(1 - \ln(\sum_{j=0}^{l} \frac{(1 - \epsilon \lambda_i)^j}{j!})). \quad (79)$$

It then follows from (78) and (79) that

$$|\lambda_i(L) - \lambda_i(L')| = \left|\lambda_i - \frac{1}{\epsilon}(1 - \ln(\sum_{j=0}^{l} \frac{(1 - \epsilon \lambda_i)^j}{j!}))\right|. \quad (80)$$

Let $E_2$ denote an upper bound on approximation error resulted from approximating the exponential of $L$ with a finite number of terms in its Taylor expansion. Since the error given in (80) is maximized when $\lambda_i = 0$, $E_2$ is defined as follows

$$E_2 = \frac{1}{\epsilon}(1 - \ln(\sum_{j=0}^{l} \frac{1}{j!})). \quad (81)$$

where

$$|\lambda_i(L) - \lambda_i(L')| \leq E_2. \quad (82)$$

According to the triangle inequality, one can conclude that

$$|\hat{\lambda}(L) - \hat{\lambda}(L'')| \leq |\hat{\lambda}(L) - \hat{\lambda}(L')| + |\hat{\lambda}(L') - \hat{\lambda}(L'')|. \quad (83)$$

It then follows from (77) and (82) that the approximation error of the GPI algorithm is upper-bounded as follows

$$|\hat{\lambda}(L) - \hat{\lambda}(L'')| \leq E_1 + E_2. \quad (84)$$

This concludes the proof.

**Remark 3:** It is to be noted that the proposed GPI algorithm has two sources of discrepancy which contribute to the approximation error of the GAC measure. The first one is related to the fact that the Krylov subspace method is an approximation technique, and does not provide the exact solution. This error is represented by $E_1$. The second one is a result of using a finite number of terms in the Taylor expansion of $e^{L}$, which is given by $E_2$. To ensure that the overall approximation error is upper-bounded by a positive constant $\delta_0$, where $\delta_0 > \rho_0$, the following inequality should hold

$$E_1 + E_2 \leq \delta_0. \quad (85)$$

The value of $l$ (the number of terms in the Taylor expansion used to approximate $e^{L}$) can then be chosen sufficiently large to satisfy the inequality (85).

**Remark 4:** To guarantee that the GPI algorithm correctly identifies the eigenvalue (or a pair of complex conjugate eigenvalues) of $L$ representing the GAC measure of a network, it suffices to choose a small enough value for $E_1$ such that the following inequality holds

$$\min_{\lambda_i(L) \in \Lambda(L), \lambda_i(L) \neq 0, \Re(\lambda_i(L)) \neq \Re(\lambda_i(L))} |\Re(\lambda_i(L)) - \hat{\lambda}(L)| > E_1. \quad (86)$$

This implies that the density of the eigenvalues, especially in the vicinity of the eigenvalue (or the pair of complex conjugate eigenvalues) representing the GAC of a network will affect the accuracy of the GPI algorithm.

**V. Simulation Results**

**Example 3:** The efficacy of Algorithm 1 in approximating the GAC of the network and the right eigenvector associated with the eigenvalue representing the GAC measure is demonstrated in this section. Consider a network composed of six nodes with a digraph $G = (\mathcal{V}, \mathcal{E}, P)$ whose weight matrix $P = [p_{ij}]$ is given by

$$P = \begin{bmatrix}
0 & 0.1 & 0.3 & 0.8 & 0.9 & 0.8 \\
0.2 & 0 & 1 & 0.1 & 0.5 & 0.1 \\
0.1 & 0.9 & 0 & 0.7 & 0.9 & 0.1 \\
1 & 0.5 & 0.6 & 0 & 0.4 & 0.6 \\
0 & 0.2 & 0.3 & 0.1 & 0 & 1 \\
0.5 & 0.4 & 0.6 & 0.1 & 0.1 & 0 \\
\end{bmatrix}. \quad (87)$$

Moreover, $0 < p_{ij} \leq 1$ for any $(j, i) \in \mathcal{E}$. For this network, $\hat{\lambda}(L) = 2.1916$ represents the smallest nonzero real part of the eigenvalues of $L$ which corresponds to a pair of complex conjugate eigenvalues $\lambda_{2.3}(L) = 2.1916 \pm j0.4313$. Consider $m = 3$, $l = 5$, and $\rho_0 = 10^{-3}$ as the parameters used in the proposed centralized GPI algorithm. Fig. 5 depicts the approximate GAC measure versus the iteration number for this example. The right eigenvector corresponding to $\hat{\lambda}(L)$

![Fig. 5. The approximate GAC measure versus step number in Example 3.](image-url)
VI. Conclusions

The notion of generalized algebraic connectivity is introduced in this paper as a novel connectivity measure for weighted asymmetric networks. This measure represents the expected asymptotic convergence rate of cooperative algorithms used to control the network, and is, in fact, an extension of the existing notion of algebraic connectivity for symmetric networks based on the Laplacian of the network graph. An analytical formulation is provided for the proposed measure using a novel matrix transformation. The generalized power iteration algorithm is then introduced to compute this measure in a centralized fashion based on the Krylov subspace approximation method and Gram-Schmidt orthonormalization procedure. The efficacy of the proposed algorithm is subsequently verified by simulations. The authors are currently developing a distributed version of the algorithm provided in this paper, as an extension of the results to more practical settings.

References


