ELEC361: Signals And Systems

Topic 3:
Fourier Series (FS)

- Introduction to frequency analysis of signals
- Fourier series of CT periodic signals
- Signal Symmetry and CT Fourier Series
- Properties of CT Fourier series
- Convergence of the CT Fourier series
- Fourier Series of DT periodic signals
- Properties of DT Fourier series
- Response of LTI systems to complex exponential
- Summary
- Appendix:
  - Applications (not in the exam)

Figures and examples in these course slides are taken from the following sources:
Signal Representation

- Time-domain representation
  - Waveform based
  - Periodic / non-periodic signals
- Frequency-domain representation
  - Periodic signals
  - Sinusoidal signals
  - Frequency analysis for periodic signals
  - Concepts of frequency, bandwidth, filtering
Waveform Representation

- Waveform representation
- Plot of the signal value vs. time
  - Sound amplitude, temperature reading, stock price, ..
- Mathematical representation: $x(t)$
  - $x$: variable value
  - $T$: independent variable
Sample Speech Waveform

Entire waveform

```matlab
» [y,fs]=wavread('morning.wav');
» sound(y,fs);
» figure; plot(y);
» x=y(10000:25000);plot(x);
```

Blown-up of a section.

```matlab
» figure; plot(x);
» axis([2000,3000,-0.1,0.08]);
```

Signal within each short time interval is periodic
Period depends on the vowel being spoken
Sample Music Waveform

Entire waveform

```matlab
[y,fs]=wavread('sc01_L.wav');
sound(y,fs);
figure; plot(y);
```

Blown-up of a section

```matlab
v=axis;
axis([1.1e4,1.2e4,-.2,.2])
```

Music typically has more periodic structure than speech. Structure depends on the note being played.
Sinusoidal Signals

\[ x(t) = A \cos(2\pi f_0 t + \phi) \]
- \( f_0 \): frequency (cycles/second)
- \( T_0 = 1/f_0 \): period
- \( A \): Amplitude
- \( \phi \): Phase (time shift)

- Sinusoidal signals: important because they can be used to synthesize any signal
  - An arbitrary signal can be expressed as a sum of many sinusoidal signals with different frequencies, amplitudes and phases
  - **Phase shift**: how much the max. of the sinusoidal signal is shifted away from \( t=0 \)
- Music notes are essentially sinusoids at different frequencies
Complex Exponential Signals

- Complex Number:
  Cartesian representation: \( z = a + jb \)
  Magnitude of \( z \) is \( |z| = \sqrt{a^2 + b^2} \)
  Phase of \( z \) is \( \phi = \angle z = \tan^{-1} \frac{b}{a} \)
  Polar representation: \( z = |z|e^{j\phi} = |z| \cos \phi + j|z| \sin \phi \)

- Complex Exponential Signal:
  \( x(t) = |z|e^{(j\omega_0t+\theta)} = |z| \cos(\omega_0t + \theta) + j|z| \sin(\omega_0t + \theta) \)
  \( \theta \) is the phase shift (initial phase)

- Complex Conjugate:
  \( z^* = a - jb \)
  Note: \( (z + z^*) \) and \( (zz^*) \) are real

- Euler Formula:
  \( e^\theta + e^{-\theta} = 2 \cos(\theta) \) \( e^\theta - e^{-\theta} = j2\sin(\theta) \)
Real and Complex Sinusoids

Euler’s formulae:

\[ \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \]

\[ \sin(x) = \frac{e^{ix} - e^{-ix}}{j2} \]
Periodic CT Signals

- A CT $x(t)$ signal is periodic if there is a positive value $T$ for which $x(t) = x(t + T)$, $\forall t$

- Period $T$ of $x(t)$: The interval on which $x(t)$ repeats
- Fundamental period $T_0$: the smallest such repetition interval $T_0 = 1/f_0$
- Fundamental period: the smallest positive value for which the equation above holds
- **Example**: $x(t) = \cos(4\pi t)$; $T=1/2$; $T_0=1/4$
- **Harmonic frequencies of** $x(t)$: $kf_0$, $k$ is integer,

- **Example**: $x(t) = 2\cos\left(\frac{t}{2} + \frac{\pi}{6}\right)$ is periodic with $T_0 = 4\pi$
Sums of CT periodic signals

- The period of the sum of CT periodic functions is the *least common multiple* of the periods of the individual functions summed.
- If the least common multiple is infinite, the sum is aperiodic.
Periodic DT Signals

- A DT $x[n]$ signal is periodic with period $N$ where $N$ is a positive integer if
  $$x[n] = x[n + N], \quad \forall n.$$  
- The fundamental period $N_0$ of $x[n]$ is the smallest positive value of $N$ for which the equation holds.
- Example:
  $$x[n] = e^{j(\frac{3\pi}{4})n}$$
  is periodic with fundamental period $N_0 = 8$. 
What is frequency of an arbitrary signal?

- Sinusoidal signals have a distinct (unique) frequency.
- An arbitrary signal $x(t)$ does not have a unique frequency.
- $x(t)$ can be decomposed into many sinusoidal signals with **different** frequencies, each with **different** magnitude and phase.
- **Spectrum** of $x(t)$: the plot of the magnitudes and phases of different frequency components.
- **Fourier analysis**: find spectrum for signals.
- **Bandwidth** of $x(t)$: the spread of the frequency components with significant energy existing in a signal.
Frequency content in signals

Lowpass

Highpass

Bandpass
Frequency content in signals

- A constant: only zero frequency component (DC component)
- A sinusoid: Contain only a single frequency component
- Periodic signals: Contain the fundamental frequency and harmonics: Line spectrum
- Slowly varying: contain low frequency only
- Fast varying: contain very high frequency
- Sharp transition: contain from low to high frequency
- Music: contain both slowly varying and fast varying components, wide bandwidth
Transforming Signals

- $X(t)$ is the signal representation in time domain
- Often we transform signals in a different domain
- Fourier analysis allows us to view signals in the frequency domain
- In the frequency domain we examine which frequencies are present in the signal
- Frequency domain techniques reveal things about the signal that are difficult to see otherwise in the time domain
Fourier representation of signals

- The study of signals and systems using sinusoidal representations is termed Fourier analysis, after Joseph Fourier (1768-1830).
- The development of Fourier analysis has a long history involving a great many individuals and the investigation of many different physical phenomena, such as the motion of a vibrating string, the phenomenon of heat propagation and diffusion.
- Fourier methods have widespread application beyond signals and systems, being used in every branch of engineering and science.
- The theory of integration, point-set topology, and eigenfunction expansions are just a few examples of topics in mathematics that have their roots in the analysis of Fourier series and integrals.
Fourier representation of signals: Types of signals

Periodic

- Analog
  - Continuous-Time Continuous-Value Signal
  - Continuous-Time Discrete-Value Signal
  - Continuous-Time Random Signal
  - Noise

- Digital
  - Discrete-Time Continuous-Value Signal
  - Discrete-Time Discrete-Value Signal
  - Noisy Digital Signal

Non-periodic
Fourier representation of signals:
Continuous-Value / Continuous-Time Signals

- All continuous signals are CT but not all CT signals are continuous
Fourier representation of signals: Types of signals

- Continuous (analog)
  - Periodic
  - Non-periodic

- Discrete (digital)
  - Periodic
  - Non-periodic

Analysis Tool

Fourier Series
Fourier Transform
Discrete Fourier Series
Discrete-time Fourier Transform & Z-transform
Fourier representation of signals

- **Four** distinct Fourier representations:
  - Each applicable to a different class of signals
  - Determined by the periodicity properties of the signal and whether the signal is discrete or continuous in time
- A Fourier representation is unique, i.e., no two same signals in time domain give the same function in frequency domain
# Overview of Fourier Analysis Methods

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**Discrete in Time**

- **CT Fourier Series** (DT → DT)
  \[ a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega t} \; dt \]

- **Inverse CT Fourier Series** (DT → CT)
  \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \]

**Periodic in Frequency**

- **CT Fourier Transform** (CT → CT)
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \; dt \]

- **Inverse CT Fourier Transform** (CT → CT)
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \; d\omega \]

**Continuous in Time**

- **CT Fourier Series** (CT - \( P_T \) → DT)
  \[ a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega t} \; dt \]

- **Inverse CT Fourier Series** (DT → CT)
  \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \]
Fourier representation: Periodic Signals

- Fourier Series Synthesis (inverser transform)
  \[ x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_0 kt + \phi_k) \] (single sided, for real signal only)
  \[ = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \] (double sided, for both real and complex)

- Fourier Series Analysis (forward transform)
  \[ X[k] = a_k = \frac{1}{T} \int_{0}^{T} x(t) e^{-j\omega_0 kt} dt; \quad k = 0, \pm 1, \pm 2, \ldots \]
  \[ a_k \text{ is, in general, a complex number} \]

- For real signals: \[ a_k = a_{-k}^* \quad |a_k| = |a_{-k}| \] (symmetric spectrum)
  \[ R(z) = \frac{z + z^*}{2} \]
FS: Example

\[ x(t) = 10 + 14 \cos(200\pi t - \frac{\pi}{3}) + 8 \cos(500\pi t + \frac{\pi}{2}) \]

\[ x(t) = 10 + 7e^{-j\frac{\pi}{3}}e^{j2\pi 100t} + 7e^{j\frac{\pi}{3}}e^{-j2\pi 100t} \]

\[ + 4e^{j\frac{\pi}{2}}e^{j2\pi 250t} + 4e^{-j\frac{\pi}{2}}e^{-j2\pi 250t} \]

\[ \Rightarrow x(t) = \text{Constant Component} + \text{Non - constant Components} \]
Concept of Fourier analysis
Concept of Fourier analysis
Approximation of Periodic Signals by Sinusoids

- Any periodic signal can be approximated by a sum of many sinusoids at harmonic frequencies of the signal \((kf_0)\) with appropriate amplitude and phase.
- The more harmonic components are added, the more accurate the approximation becomes.
- Instead of using sinusoidal signals, mathematically, we can use the complex exponential functions with both positive and negative harmonic frequencies.
Approximation of Periodic Signals by Sum of Sinusoids

\[ x(t) = \sum_{k=0}^{\infty} A_k \cos(2\pi kf_0 t) \]

2 sinusoids: 1\text{st} and 3\text{d} harmonics

4 sinusoids: 1, 3, 5, 7 harmonics
Fourier representation: Periodic CT Signals

- The Fourier series representation, $x_F(t)$, of a signal, $x(t)$, over a time, $t_0 < t < t_0 + T_F$, is

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(kf_F)t}$$

where $X[k]$ is the harmonic function, $k$ is the harmonic number and $f_F = 1/T_F$

- The harmonic function or Fourier series coefficient can be found from the signal as

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(kf_F)t} dt$$

- The signal and its harmonic function form a Fourier series pair indicated by $x(t) \xrightarrow{FS} X[k]$
Example: Fourier Series of Square Wave

\[ x(t) = \begin{cases} 
1, & |t| < T \_1 \\
0, & |t| > T \_1 
\end{cases} \]

Square Signal

The Fourier series analysis:

\[ a_k = \begin{cases} 
\frac{2}{j \pi k}, & k = \pm1,3,5,\ldots \\
0, & k = 0, \pm2,4,\ldots 
\end{cases} \]
Example: Spectrum of Square Wave

- Each line corresponds to one harmonic frequency. The line magnitude (height) indicates the contribution of that frequency to the signal.

\[ a_k = \begin{cases} \frac{4}{\pi k} & k = 1, 3, 5, \ldots \\ 0 & k = 0, 2, 4, \ldots \end{cases} \]

- The line magnitude drops exponentially, which is not very fast. The very sharp transition in square waves calls for very high frequency sinusoids to synthesize.

- Only the positive frequency side is drawn on the left (single sided spectrum), with twice the magnitude of the double sided spectrum.
Negative Frequency?

This signal is obviously a sinusoid. How is it described mathematically?

It could be described by:

\[ x(t) = A \cos \left( \frac{2\pi f_0 t}{T_0} \right) = A \cos (2\pi f_0 t) \]

But it could also be described by:

\[ x(t) = A \cos (2\pi (-f_0) t) \]
Negative Frequency?

$x(t)$ could also be described by

$$x(t) = A \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

or

$$x(t) = A_1 \cos(2\pi f_0 t) + A_2 \cos(2\pi (-f_0) t), \quad A_1 + A_2 = A$$

and probably in a few other different-looking ways. So who is to say whether the frequency is positive or negative? For the purposes of signal analysis, it does not matter.
Why Frequency Domain Representation of signals?

- Shows the frequency composition of the signal
- Change the magnitude of any frequency component arbitrarily by a **filtering** operation
  - Lowpass -> smoothing, noise removal
  - Highpass -> edge/transition detection
  - High emphasis -> edge enhancement
- Shift the central frequency by modulation
  - A core technique for **communication**, which uses modulation to multiplex many signals into a single composite signal, to be carried over the same physical medium
Why Frequency Domain Representation of signals?

Typical Filtering applied to $x(t)$:
- Lowpass -> smoothing, noise removal
- Highpass -> edge/transition detection
- Bandpass -> Retain only a certain frequency range
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Fourier series of CT periodic signals

- Consider the following continuous-time complex exponentials:
  \[ \phi_k(t) = e^{j\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots \]

- \( T_0 \) is the period of all of these exponentials and it can be easily verified that the fundamental period is equal to \( 2\pi/k\omega_0 \)

\[
\phi_k(t + T_0) = e^{jk\omega_0 (t+T_0)} = e^{jk\omega_0 t + \frac{2\pi}{k\omega_0}} = e^{jk\omega_0 t + j2\pi} = e^{jk\omega_0 t}.
\]

- Any linear combination of \( \phi_k(t) \) is also periodic with period \( T_0 \)

\[
x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (4.1)
\]
Fourier series of CT periodic signals

- $\phi_1(t)$ and $\phi_{-1}(t)$ fundamental components or the first harmonic components
  - The corresponding fundamental frequency is $\omega_0$

- **Fourier series representation** of a periodic signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$  \hspace{1cm} (4.1)
Fourier series of a CT periodic signal

- Consider the periodic signal $x(t)$ given by (4.1). Multiplying both sides of (4.1) by $e^{-jn\omega_0 t}$ we will have:

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\Rightarrow \int_{0}^{T} x(t)e^{-jn\omega_0 t} \, dt = \int_{0}^{T} \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \, dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left[ \int_{0}^{T} e^{j(k-n)\omega_0 t} \, dt \right]$$

(4.3)

- On the other hand:

$$\int_{0}^{T} e^{j(k-n)\omega_0 t} \, dt = \int_{0}^{T} \cos((k-n)\omega_0 t) \, dt + j \int_{0}^{T} \sin((k-n)\omega_0 t) \, dt$$

$$= \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

(4.4)
Fourier series of a CT periodic signal

- From (4.3) and (4.4) it can be concluded that:

\[ a_n = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 t} \, dt \]

- The coefficient \( a_0 = \frac{1}{T} \int_0^T x(t) \, dt \) represents the average value of \( x(t) \) over one period.

- In general, we can write the following pair equations which are called the Fourier series of a periodic continuous-time signal:

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \]

\[ a_k = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 t} \, dt, \quad \omega_0 = \frac{2\pi}{T} \] \hspace{1cm} (4.5)

- \( \omega_0 \) is the fundamental frequency and \( T \) is the fundamental period of \( x(t) \).

- The set of coefficients \( \{a_k\} \) are called the Fourier series coefficients of the spectral coefficients of \( x(t) \).

- The Fourier series of a periodic signal \( x(t) \) is also denoted by:

\[ x(t) \xrightarrow{\text{FS}} a_k \]
Fourier series of a CT periodic signal: Example 4.1

- **Example 4.1**: Find the Fourier series coefficients for the following periodic signal:

\[ x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4}) \]

- **Solution**: We have:

\[
x(t) = 1 + \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2} [e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}]
\]

\[
= \frac{1}{a_0} + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)e^{-j2\omega_0 t}
\]

This implies that:

\[
a_0 = 1, \quad a_1 = 1 - \frac{j}{2}, \quad a_{-1} = 1 + \frac{j}{2}, \quad a_2 = \frac{\sqrt{2}}{4} (1 + j), \quad a_{-2} = \frac{\sqrt{2}}{4} (1 - j), \quad a_k = 0, \quad |k| > 2
\]
The Fourier series coefficients are shown in the Figures.

Note that the Fourier coefficients are complex numbers in general.

Thus one should use two figures to demonstrate them completely: show
- real and imaginary parts or
- magnitude and angle.
If $x(t)$ is **real**

$$x(t) = x^*(t) = \sum_{k=-\infty}^{+\infty} a_k e^{-j k \omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k^* e^{j k \omega_0 t}$$

$$\Rightarrow a_k = a_k^*, \quad \text{or:} \quad a_k^* = a_{-k}$$

This means that

$$x(t) = a_0 + \sum_{k=1}^{+\infty} \left[ a_k e^{j k \omega_0 t} + a_{-k} e^{-j k \omega_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{+\infty} \left[ a_k e^{j k \omega_0 t} + a_k^* e^{-j k \omega_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{+\infty} 2 \text{Re}\{a_k e^{j k \omega_0 t}\} \quad (4.2)$$
Fourier series of CT REAL periodic signals

Let \( a_k \) be denoted by \( a_k = A_k e^{j \theta_k} \), where \( \theta_k \) and \( A_k \) are both real values. Then, from (4.2) we will have:

\[
x(t) = a_0 + \sum_{k=1}^{+\infty} 2 \text{Re} \{A_k e^{j(k \omega_0 t + \theta_k)}\}
\]

\[
= a_0 + 2 \sum_{k=1}^{+\infty} A_k \cos(k \omega_0 t + \theta_k).
\]

Alternatively let \( a_k \) be denoted by \( a_k = B_k + j C_k \), where \( B_k \) and \( C_k \) are both real values. Then, from (4.2) we will have:

\[
x(t) = a_0 + 2 \sum_{k=1}^{+\infty} [B_k \cos(k \omega_0 t) - C_k \sin(k \omega_0 t)].
\]
Functions of the form $f(x) = \frac{\sin(ax)}{bx}$ are very frequently used in signals and systems. In general, the function $\frac{\sin(\pi x)}{\pi x}$ is referred to as sinc function and can be expressed as follows:

$$x(t) = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi} \text{sinc} \left( \frac{Wt}{\pi} \right)$$
Fourier series of a CT periodic signal: Example 4.2

Find the Fourier series coefficients of the following periodic signal:

\[ x(t) = \begin{cases} 
1 & |t| < T_1 \\
0 & T_1 < |t| < \frac{T}{2} 
\end{cases}, \quad x(t + T) = x(t), \forall t. \]

(The signal \( x(t) \) is depicted in Figure 4.2).
Fourier series of a CT periodic signal: Example 4.2

Solution: From (4.5), we have:

\[ a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-j k \omega_0 t} \, dt = -\frac{1}{j k \omega_0 T} e^{-j k \omega_0 T} \bigg|_{-T_1}^{T_1} \]

\[ = \frac{2}{k \omega_0 T} \left[ \frac{e^{jk \omega_0 T_1} - e^{-jk \omega_0 T_1}}{2j} \right] \]

\[ = \frac{\sin(k \omega_0 T_1)}{k \pi}, \quad k \neq 0 \]

\[ a_0 = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \]

- The Fourier series coefficients are shown in Figures for \( T=4T_1 \) and \( T=16T_1 \).
- Note that the Fourier series coefficients for this particular example are real.
Figure 4.3: The Fourier series coefficients for Example 4.2. (a) $T = 4T_1$; (b) $T = 16T_1$. 
Inverse CT Fourier Series:
Example: Magnitude and Phase Spectra of the harmonic function $X[k]$
Inverse CT Fourier series: Example

- The CT Fourier Series representation $x_F(t)$ of the above cosine signal $X[k]$

- $x(t)$ is odd
- The discontinuities make $X[k]$ have significant higher harmonic content
Consider the two (different) transfer functions,
\[ H_1(f) = \frac{1}{j2\pi f + 1} \quad \text{and} \quad H_2(f) = \frac{30}{30 - 4\pi^2 f^2 + j62\pi f} \]

When plotted on this scale, these magnitude frequency response plots are indistinguishable.
Note: Log-Magnitude Frequency Response Plots

When the magnitude frequency responses are plotted on a logarithmic scale the difference is visible
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Effect of Signal Symmetry on CT Fourier Series

- Unnecessary work (and corresponding sources of errors) in determining Fourier coefficients of periodic signals can be avoided if the signals possess any type of symmetry.

- The important types of symmetry are:

  1. even symmetry, \( x(t) = x(-t) \),
  2. odd symmetry, \( x(t) = -x(-t) \),
  3. half-wave odd symmetry, \( x(t) = -x(t + \frac{T}{2}) \)
Effect of Signal Symmetry on CT Fourier Series

- Recognizing the existence of one or more of these symmetries simplifies the computation of the Fourier-series coefficients.

- For example, the Fourier series of an even signal \( x(t) \) having period \( T \) is a “Fourier cosine series,”

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n \pi t}{T}
\]

with coefficients

\[
a_0 = \frac{2}{T} \int_{0}^{T/2} x(t) \, dt, \quad \text{and} \quad a_n = \frac{4}{T} \int_{0}^{T/2} x(t) \cos \frac{2n \pi t}{T} \, dt
\]

- The Fourier series of an odd signal \( x(t) \) having period \( T \) is a “Fourier sine series,”

\[
x(t) = \sum_{n=1}^{\infty} b_n \sin \frac{2n \pi t}{T}
\]

with coefficients

\[
b_n = \frac{4}{T} \int_{0}^{T/2} x(t) \sin \frac{2n \pi t}{T} \, dt
\]
Effect of Signal Symmetry on CT Fourier Series

- It can be shown that the CT harmonic function, $X[k]$, of any real-valued function, $x(t)$, has the property

  $$X[k] = X^*[-k]$$

- The magnitude of the harmonic function is an even function and

- the phase of $X[k]$ is an odd function
Effect of Signal Symmetry on CT Fourier Series

- For an even function
  - The harmonic function, $X[k]$, is purely real
  - The sine harmonic function is zero

- For an odd function
  - The harmonic function, $X[k]$, is purely imaginary
  - The cosine harmonic function is zero
Effect of Signal Symmetry on CT Fourier Series

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<tbody>
<tr>
<td>Even</td>
<td>$a_0 \neq 0$</td>
<td>$a_n \neq 0$</td>
<td>$b_n = 0$</td>
<td>Integrate over $T/2$ only, and multiply the coefficients by 2.</td>
</tr>
<tr>
<td>Odd</td>
<td>$a_0 = 0$</td>
<td>$a_n = 0$</td>
<td>$b_n \neq 0$</td>
<td>Integrate over $T/2$ only, and multiply the coefficients by 2.</td>
</tr>
<tr>
<td>Half-wave odd</td>
<td>$a_0 = 0$</td>
<td>$a_{2n} = 0$</td>
<td>$b_{2n} = 0$</td>
<td>Integrate over $T/2$ only, and multiply the coefficients by 2.</td>
</tr>
<tr>
<td></td>
<td>$a_{2n+1} \neq 0$</td>
<td>$b_{2n+1} \neq 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Consider the signal $x(t)$

$$x(t) = \begin{cases} 
A - \frac{4A}{T} t, & 0 < t < T/2 \\
\frac{4A}{T} t - 3A, & T/2 < t < T 
\end{cases}$$

Notice that $x(t)$ is both an even and a half-wave odd signal. Therefore, $a_0 = 0$, $b_n = 0$, and we expect to have no even harmonics. Computing $a_n$, we obtain

$$a_n = \frac{4}{T} \int_0^{T/2} \left( A - \frac{4A}{T} t \right) \cos \frac{2n\pi t}{T} dt$$

$$= \frac{4A}{(n\pi)^2} (1 - \cos(n\pi)), \quad n \neq 0$$

$$= \begin{cases} 
0, & n \text{ even} \\
\frac{8A}{(n\pi)^2}, & n \text{ odd} 
\end{cases}$$

Observe that $a_0$, which corresponds to the dc term (the zero harmonic), is zero because the area under one period of $x(t)$ evaluates to zero.
Outline

- Introduction to frequency analysis
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- Summary
Properties of CT Fourier series

- The properties are useful in determining the Fourier series or inverse Fourier series.
- They help to represent a given signal in terms of operations (e.g., convolution, differentiation, shift) on another signal for which the Fourier series is known.
- Operations on \( \{x(t)\} \leftrightarrow \) Operations on \( \{X[k]\} \)
- Help find analytical solutions to Fourier Series problems of complex signals.
- Example:
  
  \[ \text{FS}\{y(t) = a^t u(t - 5)\} \rightarrow \text{delay and multiplication} \]
Properties of CT Fourier series

- Let \( x(t) \): have a fundamental period \( T_{0x} \)
- Let \( y(t) \): have a fundamental period \( T_{0y} \)
- Let \( X[k] = a_k \) and \( Y[k] = b_k \)
  - The Fourier Series harmonic functions each using the fundamental period \( T_F \) as the representation time
- In the Fourier series properties which follow:
  - Assume the two fundamental periods are the same
    \( T = T_{0x} = T_{0y} \) (unless otherwise stated)
- The following properties can easily been shown using equation (4.5) for Fourier series
Properties of CT Fourier series

Linearity:

\[ x(t) \xrightarrow{FS} a_k, \quad y(t) \xrightarrow{FS} b_k, \quad \text{Period } T \]

\[ \Rightarrow Ax(t) + By(t) \xrightarrow{FS} Aa_k + Bb_k, \quad \text{Period } T \]
Properties of CT Fourier series

Time Shifting

\[ x(t) \xrightarrow{\text{FS}} a_k = X[k] \quad \text{Period } T \]

\[ x(t - t_0) \xrightarrow{\text{FS}} e^{-j2\pi(kt_0)} X[k] \]

\[ x(t - t_0) \xrightarrow{\text{FS}} e^{-j(k\omega_0)t_0} X[k] \]

Does not alter the FS Coefficients
Properties of CT Fourier series

**Shift in Time**

- If $x(t)$ has the Fourier-series coefficients $c_n$, then the signal $x(t - \tau)$ has coefficients $d_n$, where
  \[
  d_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t - \tau) \exp[-jn\omega_0 t] \, dt
  \]
  
  \[
  = \exp[-j\omega_0 \tau] \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\sigma) \exp[-jn\omega_0 \sigma] \, d\sigma
  \]
  
  \[
  = c_n \exp[-jn\omega_0 \tau]
  \]

- Thus, if the Fourier-series representation of a periodic signal $x(t)$ is known relative to one origin, the representation relative to another origin shifted by $\tau$ is obtained by adding the phase shift $n\omega_0 \tau$ to the phase of the Fourier coefficients of $x(t)$.
Properties of CT Fourier series

Time reversal:

\[ x(t) \xrightarrow{\text{FS}} a_k, \quad \text{Period } T \]

\[ \Rightarrow x(-t) \xrightarrow{\text{FS}} a_{-k}, \quad \text{Period } T \]
Properties of CT Fourier series

- **Time scaling:**

  \[ x(t) \overset{FS}{\longleftrightarrow} a_k, \quad \text{Period } T \]

  \[ \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha \omega_0)t}, \quad \text{Period } \frac{T}{\alpha} \]

  \[ \Rightarrow x(\alpha t) \overset{FS}{\leftrightarrow} a_k, \quad \text{Period } \frac{T}{\alpha} \]

  *Change the period of \( x(t) \) but not \( a_k \)

- **Multiplication:**

  \[ x(t) \overset{FS}{\leftrightarrow} a_k, \quad y(t) \overset{FS}{\leftrightarrow} b_k, \quad \text{Period } T \]

  \[ \Rightarrow x(t)y(t) \overset{FS}{\leftrightarrow} \sum_{l=-\infty}^{+\infty} a_l b_{k-l}, \quad \text{Period } T \]

  *Periodic*
Properties of CT Fourier series

- Conjugation and conjugate symmetry:
  \[ x(t) \leftrightarrow_{FS} a_k, \text{ Period } T \]
  \[ \Rightarrow x^* (t) \leftrightarrow_{FS} a^*_{-k}, \text{ Period } T \]

- For real signals we have \( x(t) = x^* (t) \), which results in the following conjugate symmetry:
  \[ a_{-k} = a^*_k \]

- For real and even signals we have \( x(t) = x^* (t) \) and \( x(t) = x(-t) \), which result in real and even Fourier coefficients:
  \[ a_k = a^*_k, \quad a_{-k} = a_k \]

- For real and odd signals we have \( x(t) = x^* (t) \) and \( x(-t) = -x(t) \), which result in purely imaginary and odd Fourier coefficients:
  \[ a_k = -a^*_k, \quad a_{-k} = -a_k \]
Properties of CT Fourier series

- Parseval’s relation:

\[
\frac{1}{T} \int_{T} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2
\]

- \( |a_k|^2 \) is in fact the average power in the \( k^{\text{th}} \) harmonic component of \( x(t) \).

- Parseval’s relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.
Properties of CT Fourier series

- Integration: For a periodic signal $x(t)$ with zero average over one period ($a_0 = 0$), we will have a finite valued periodic integral. The Fourier series coefficients of the integral will be:

$$x(t) \xrightarrow{FS} a_k, \quad \text{Period } T$$

$$\Rightarrow \int_{-\infty}^{\infty} x(\tau)d\tau \xrightarrow{FS} \frac{1}{jk\omega_0} a_k, \quad \text{Period } T$$

- Periodic convolution:

$$x(t) \xrightarrow{FS} a_k, \quad \text{Period } T$$

$$y(t) \xrightarrow{FS} b_k, \quad \text{Period } T$$

$$\Rightarrow \int_T x(\tau)y(t-\tau)d\tau \xrightarrow{FS} Ta_k b_k, \quad \text{Period } T$$

$$x(t) \rightarrow [h(t)] \rightarrow y(t)$$

$$y(t) = x(t) \ast h(t)$$

$FS$ Coef. of $y(t)$

$Ta_k b_k$
Properties of CT Fourier series

Differentiation:

\[ x(t) \xrightarrow{\text{FS}} a_k, \quad \text{Period } T \]

\[ \frac{dx(t)}{dt} \xrightarrow{\text{FS}} jk\omega_0 a_k, \quad \text{Period } T \]
Properties of CT Fourier series

Frequency Shifting (Harmonic Number Shifting)

\[ e^{j2\pi(k_0f_0)t} x(t) \xrightarrow{\text{FS}} X[k - k_0] \]

\[ e^{j(k_0\omega_0)t} x(t) \xrightarrow{\text{FS}} X[k - k_0] \]

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential
Properties of CT Fourier series: Example 5.1

Find the Fourier series representation of the following periodic signal:

\[ x(t) \quad \text{even and real} \quad \text{impulse train} \]

- **Solution:** We have:

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \]

\[ a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\omega_k t} \, dt \]

\[ = \frac{1}{T} \quad \text{(Using the sifting property of impulse)} \]
Properties of CT Fourier series: Example 5.2

- Find the Fourier series representation of the following periodic signal:

- Solution: We have:

\[ y(t) = x(t + T_1) - x(t - T_1) \]

\[ \Rightarrow y(t) \xlongleftarrow{\text{FS}} e^{jk\omega_0 T_1}a_k - e^{-jk\omega_0 T_1}a_k \]

\[ \xlongleftarrow{\text{FS}} \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] \]

\[ \xlongleftarrow{\text{FS}} \frac{2j \sin(k\omega_0 T_1)}{T} \]
Properties of CT Fourier series: Example 5.2

- It can be easily verified that the Fourier series coefficients of the signal $y(t)$ given in Example 5.2 is in fact equal to $jk\omega_0$ times the Fourier series coefficients of the following signal which was given in Example 4.2:

![Diagram of signal z(t)]
Properties of CT Fourier series: Example

- Compute the average power of: \( x(t) = A \cos(\omega_0 t) \)

Need to compute: \( \int_{T_0}^T [A \cos(\omega_0 t)]^2 \, dt \)

We know that the Fourier series of \( [A \cos(\omega_0 t)] \) is:

\[
X(1) = A/2, \quad X(-1) = A/2, \quad \text{and by Parseval's theorem:}
\]

\[
\sum_{k=-1}^{1} |X(k)|^2 = [X(1)]^2 + [X(-1)]^2 + [X(0)]^2 = \frac{A^2}{4} + \frac{A^2}{4} + 0 = \frac{A^2}{2}
\]

\[
X(0) = \int_{T_0}^T \cos(\omega_0 t) \, dt = 0
\]
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Convergence of the CT Fourier series

- The Fourier series representation of a periodic signal $x(t)$ converges to $x(t)$ if the Dirichlet conditions are satisfied.
- Three Dirichlet conditions are as follows:
  1. Over any period, $x(t)$ must be absolutely integrable. 

$$\int_{T} |x(t)| dt < \infty.$$ 

For example, the following signal does not satisfy this condition.

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

$$x(t + 1) = x(t), \quad \forall t.$$
Convergence of the CT Fourier series

2. $x(t)$ must have a finite number of maxima and minima in one period

For example, the following signal meets Condition 1, but not Condition 2

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

$$x(t + 1) = x(t), \quad \forall t.$$
Convergence of the CT Fourier series

3. $x(t)$ must have a finite number of discontinuities, all of finite size, in one period

For example, the following signal violates Condition 3

$$x(t) = \begin{cases} 
1 & 0 \leq t < 4 \\
2 & 4 \leq t < 6 \\
1 & 6 \leq t < 7 \\
\vdots & \vdots 
\end{cases}$$

$$x(t+1) = x(t), \forall t.$$
Convergence of the CT Fourier series

- Every continuous periodic signal has an FS representation.
- Many not continuous signals have an FS representation.
- If a signal $x(t)$ satisfies the Dirichlet conditions and is not continuous, then the Fourier series converges to the midpoint of the left and right limits of $x(t)$ at each discontinuity.
- Almost all physical periodic signals encountered in engineering practice, including all of the signals with which we will be concerned, satisfy the Dirichlet conditions.
Convergence of the CT Fourier series: Summary

For the Fourier series to converge, the signal $x(t)$ must possess the following properties, which are known as the Dirichlet conditions, over any period:

1. $x(t)$ is absolutely integrable: that is,

   $$\int_{h}^{h+T} |x(t)| \, dt < \infty$$

2. $x(t)$ has only a finite number of maxima and minima.

3. The number of discontinuities in $x(t)$ must be finite.

These conditions are sufficient, but not necessary. Thus if a signal $x(t)$ satisfies the Dirichlet conditions, then the corresponding Fourier series is convergent and its sum is $x(t)$, except at any point $t_0$ at which $x(t)$ is discontinuous. At the points of discontinuity, the sum of the series is the average of the left- and right-hand limits of $x(t)$ at $t_0$; that is,

$$x(t_0) = \frac{1}{2} [x(t_0^+) + x(t_0^-)] \quad (3.4.1)$$
Convergence of the CT Fourier series: Continuous signals

For continuous signals, convergence is exact at every point.

A Continuous Signal

Partial CTFS Sums
Convergence of the CT Fourier series:
Discontinuous signals

For discontinuous signals convergence is exact at every point of continuity.

Discontinuous Signal
Convergence of the CT Fourier series: Gibb’s phenomenon

\[ x_N(t) = \sum_{n=-N}^{N} c_n \exp[jn\omega_0 t] \]

\[ = \sum_{n=-N}^{N} \frac{1}{T} \int_{(T)} x(\tau) \exp[-jn\omega_0 \tau] d\tau \exp[jn\omega_0 t] \]

\[ = \frac{1}{T} \int_{(T)} x(\tau) \left\{ \sum_{n=-N}^{N} \exp[jn\omega_0 (t - \tau)] \right\} d\tau \]

Let

\[ g(t - \tau) = \sum_{n=-N}^{N} \exp[jn\omega_0 (t - \tau)] = \frac{\sin\left(\frac{(N + 1)}{2} \omega_0 (t - \tau)\right)}{\sin\left(\frac{\omega_0 t - \tau}{2}\right)} \]

then

\[ x_N(t) = \frac{1}{T} \int_{(T)} x(\tau) \frac{\sin\left(\frac{(N + 1)}{2} \omega_0 (t - \tau)\right)}{\sin\left(\frac{\omega_0 t - \tau}{2}\right)} d\tau \]

Dirichlet function

N=6

\[ g^{(\omega)} \]
Convergence of the CT Fourier series: Gibbs Phenomenon

The Gibbs Phenomenon

\[ N = 199 \]
\[ N = 59 \]
\[ N = 19 \]
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The DT Fourier Series

\[ x_F[n] = \sum_{k=\langle N_F \rangle} X[k] e^{j2\pi(kF_F)n} \quad X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] e^{-j2\pi(kF_F)n} \]

where \( N_F \) is the representation time, \( F_F = \frac{1}{N_F} \), and the notation,

\[ \sum_{k=\langle N_F \rangle} \]

means a summation over any range of consecutive \( k \)'s exactly \( N_F \) in length.
The DT Fourier Series

Notice that in

\[ x_F[n] = \sum_{k=N_F}^{k} X[k] e^{j2\pi (kF_F)n} \]

the summation is over exactly one period, a finite summation. This is because of the periodicity of the complex sinusoid,

\[ e^{-j2\pi (kF_F)n} \]

in harmonic number, \( k \). That is, if \( k \) is increased by any integer multiple of \( N_F \) the complex sinusoid does not change

\[ e^{-j2\pi (kF_F)n} = e^{-j2\pi ((k+mN_F)F_F)n} \quad (m \text{ an integer}) \]

This occurs because discrete time, \( n \), is always an integer
The DT Fourier Series

In the very common case in which the representation time is taken as the fundamental period, \( N_0 \), the DTFS is

\[
x[n] = \sum_{k=\langle N_0 \rangle} X[k] e^{j2\pi(kF_0)n} \quad \text{FS} \quad X[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] e^{-j2\pi(kF_0)n}
\]

or in terms of radian frequency

\[
x[n] = \sum_{k=\langle N_0 \rangle} X[k] e^{j(k\Omega_0)n} \quad \text{FS} \quad X[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] e^{-j(k\Omega_0)n}
\]

where \( \Omega_0 = 2\pi F_0 = \frac{2\pi}{N_0} \)
Concept of DT Fourier Series

Harmonics

Real

$k = 0$

$k = 1$

$k = -1$

$k = 2$

Imaginary

$k = 0$

$k = 1$

$k = -1$

$k = 2$

Sum of Harmonics

$x[n]$ Original Signal
The DT Fourier Series

- Consider the following **discrete-time complex exponentials**: 

  \[ \phi_k[n] = e^{j k \omega_d n} = e^{j k \frac{2 \pi}{N}} , \quad k = 0, \pm 1, \pm 2, \ldots \]

  (These complex exponentials are similar to the continuous-time counterparts 
  \[ \phi_k(t) = e^{j k \omega_d t} = e^{j k \frac{2 \pi}{T}} \] 
  that we used for continuous-time periodic signals).

- These signals are all **periodic** with period \( N \) and the fundamental frequencies are 
  multiples of \( 2 \pi / N \). It can be easily seen that:

  \[ \phi_k[n + N] = e^{j k \frac{2 \pi (n+N)}{N}} = e^{j k \frac{2 \pi n}{N} + j 2 \pi} = e^{j k \frac{2 \pi}{N}} = \phi_k[n]. \]
The DT Fourier Series

- This implies that unlike continuous-time case, here we only have $N$ distinct complex exponentials. In other words, $\phi_{k+N}[n]$ is equal to $\phi_k[n]$.

- Consider now a periodic signal $x[n]$, which is a linear combination of the sequences $\phi_k[n]$ as follows:

$$x[n] = \sum_{k} a_k e^{\frac{jk2\pi n}{N}}.$$  

- Since there are only $N$ distinct complex exponentials $\phi_k[n]$, the summation will have $N$ terms only. This will be denoted by:

$$x[n] = \sum_{k=}^{<N>} a_k e^{\frac{jk2\pi n}{N}}.$$
The DT Fourier Series

- For instance, one can use the summation from $k = 0$ to $k = N - 1$ as follows:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk \frac{2\pi n}{N}}.$$

- All discrete-time periodic signals have Fourier series representation. The following pair of equations can be obtained in a way similar to the continuous-time case:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk \frac{2\pi n}{N}}$$

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk \frac{2\pi n}{N}}$$
The DT Fourier Series: Example 5.3

Consider the signal shown in the following figure.

We want to obtain the coefficients of its Fourier series.
The DT Fourier Series: Example 5.3

- Solution:

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j\frac{k(2\pi n)}{N}}. \]

- From Problem 1.54 on Page 73 of the text-book, we know that:

\[ \sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \text{For any complex number } \alpha \neq 1. \end{cases} \]

- One can modify this result as follows:

\[ \sum_{n=-N_1}^{N_1} \alpha^n = \begin{cases} 2N_1 + 1, & \alpha = 1 \\ \frac{\alpha^{-N_1} - \alpha^{N_1+1}}{1-\alpha}, & \text{For any complex number } \alpha \neq 1 \end{cases} \]
The DT Fourier Series: Example 5.3

- Using this formula with $\alpha = e^{-jk(\frac{\pi}{N})}$ and assuming that $k \neq 0, \pm N, \pm 2N, \ldots$ (for which $\alpha = 1$), we will have:

$$a_k = \frac{1}{N} \left( e^{jk\left(\frac{2\pi N_1}{N}\right)} - e^{-jk\left(\frac{2\pi (N_1+1)}{N}\right)} \right)$$

$$= \frac{1}{N} \left( e^{-jk\left(\frac{2\pi}{2N}\right)} \left[ e^{jk\left(\frac{2\pi (N_1+1/2)}{N}\right)} - e^{-jk\left(\frac{2\pi (N_1+1/2)}{N}\right)} \right] \right)$$

$$= \frac{1}{N} \left( \frac{2j \sin\left(\frac{2\pi k(N_1 + 1/2)}{N}\right)}{2j \sin\left(\frac{2\pi k}{2N}\right)} \right)$$
The DT Fourier Series: Example 5.3

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1 + 1}{N} \]

- For \( k = 0, \pm N, \pm 2N, \ldots \) we will have:

- The Fourier series coefficients for the case when \( N_1 = 2 \) and \( N = 10 \) are shown in the following figure:
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Properties of DT Fourier Series

- Linearity, time shifting, frequency shifting and time reversal properties are very similar to their continuous-time counterparts. The following properties are a little bit different though:

- Multiplication: Assume \( x[n] \) and \( y[n] \) are periodic signals with period \( T \) and the following Fourier series representations:

\[
x[n] \xrightarrow{FS} a_k,
\]

\[
y[n] \xrightarrow{FS} b_k.
\]

Then, we will have:

\[
x[n]y[n] \xrightarrow{FS} c_k = \sum_{l=-N}^{N} a_l b_{k-l}.
\]

(Note that for CT periodic signals the summation limits were from \(-\infty\) to \(+\infty\)
Properties of DT Fourier Series

- **First Difference**: Consider the signal:

  \[ x[n] \xleftarrow{\text{FS}} a_k . \]

  It can be easily seen that:

  \[ x[n] - x[n-1] \xleftarrow{\text{FS}} (1 - e^{-j(k \frac{2\pi}{N})})a_k . \]

- **Parseval’s Relation for Discrete-Time Periodic Signals**: It can be shown that:

  \[
  \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2 .
  \]

  (As an example, check the periodic square wave and its Fourier series coefficients).

- Properties of the discrete-time Fourier series are summarized in Table 3.2 in Page 221.
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Response of LTI systems to Complex Exponential

- Eigen-function of a linear operator $S$:
  - a non-zero function that returns from the operator exactly as is except for a multiplicator (or a scaling factor)

- Eigenfunction of a system $S$:
  - characteristic function of $S$

System applied on a function $x(t): S\{x(t)\} = \lambda x(t)$

$\lambda$: eigen-value (a non-null vector)

$x(t)$: eigen-function
Response of LTI systems to Complex Exponential

- Eigenfunction of an LTI system = the complex exponential $e^{j\omega_k t}$
- Any LTI system $S$ excited by a complex sinusoid responds with another complex sinusoid of the same frequency, but generally a different amplitude and phase
- The eigen-values are either real or, if complex, occur in complex conjugate pairs
Response of LTI systems to Complex Exponential

- Convolution represents:
  - The input as a linear combination of impulses
    \[ x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \]
  - The response as a linear combination of impulse responses
    \[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \]

- Fourier Series represents:
  - A periodic signal as a linear combination of complex sinusoids
    \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t} \quad \omega_k = 2\pi f_k = 2\pi k f_0 \]
Response of LTI systems to Complex Exponential: Linearity and Superposition

- If \( x(t) \) can be expressed as a sum of complex sinusoids
  - the response can be expressed as the sum of responses to complex sinusoids

\[
y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_k t} \quad \omega_k = 2\pi f_k
\]

\[
x(t) = a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} + a_3 e^{j\omega_3 t}
\]
Response of LTI systems to Complex Exponential

- Let a continuous-time LTI system be excited by a complex exponential of the form,
  \[ x(t) = e^{st} \]

- The response is the convolution of the excitation with the impulse response or
  \[ y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau \]

- The quantity
  \[ \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau \]

will later be designated the Laplace transform of the impulse response and will be an important transform method for CT system analysis.
Response of LTI systems to Complex Exponential

- CT system: \( x(t) = e^{st} = e^{(\sigma + j\omega)t} \); \( s = \sigma + j\omega \)

\[
x(t) = e^{st} \rightarrow h(t) \rightarrow y(t)
\]

\[
y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau
\]

\[
= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau
\]

\[
= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-st}d\tau
\]

- This leads to the following equation for CT LTI systems:

\[
y(t) = H(s)e^{st}
\]
Response of LTI systems to Complex Exponential

- $H(s)$ is a complex constant whose value depends on $s$ and is given by:

$$y(t) = H(s)e^{st} \quad H(s) \doteq \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau} d\tau$$

- Complex exponential $e^{st}$ are eigenfunctions of CT LTI systems
- $H(s)$ is the eigenvalue associated with the eigenfunction $e^{st}$
Response of LTI systems to Complex Exponential

- DT systems:

\[ x[n] = z^n \rightarrow h[n] \rightarrow y[n] \]

\[
y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]
= \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}
= z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}
\]

This leads to the following equation for DT LTI systems:

\[ y[n] = H(z) z^n \]
Response of LTI systems to Complex Exponential

- \( H(z) \) is a complex constant whose value depends on \( z \) and is given by:

\[
H(z) \triangleq \sum_{k=-\infty}^{+\infty} h[k] z^{-k}
\]

- Complex exponential \( z^n \) are eigenfunctions of DT LTI systems
- \( H(z) \) is the eigenvalue associated with the eigenfunction \( z^n \)
Response of LTI systems to Complex Exponential

- From superposition:

\[ x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} \rightarrow h(t) \rightarrow y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} \]

\[ x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t} \]

\[ x[n] = \sum_k a_k z_k^n \rightarrow y[n] = \sum_k a_k H(z_k) z_k^n \]
Response of LTI systems to Complex Exponential: Example 3.3

- Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by a time shift as follows:

$$x(t) \rightarrow y(t) = x(t - 3).$$

- Find the output of the system to the following inputs:

  a) $x(t) = e^{j2t}$
  b) $x(t) = \cos(4t) + \cos(7t)$
Example 3.3 - Solution

a) We have $x(t) = e^{j2t} \Rightarrow y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$.

The above expression is in fact the same as $y(t) = H(j2)e^{j2t}$, where $H(j2)$ is equal to $e^{-j6}$. This can be confirmed using

$$H(s) \triangleq \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

as follows:

$$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3)e^{-s\tau}d\tau = e^{-3s}$$

It can be easily verified that $H(s)|_{s=j} = e^{-j6}$. 
Example 3.3 - Solution

b) We have \( x(t) = \cos(4t) + \cos(7t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t} \)

\[ y(t) = H(s) x(t) \]

This leads to the following expression for the output:

\[
y(t) = \frac{1}{2} H(s)|_{s=j4} e^{j4t} + \frac{1}{2} H(s)|_{s=-j4} e^{-j4t} + \frac{1}{2} H(s)|_{s=j7} e^{j7t} \\
+ \frac{1}{2} H(s)|_{s=-j7} e^{-j7t} \\
= \frac{1}{2} e^{-j12} e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j21} e^{j7t} + \frac{1}{2} e^{j21} e^{-j7t} \\
= \frac{1}{2} e^{j4(t-3)} + \frac{1}{2} e^{-j4(t-3)} + \frac{1}{2} e^{j7(t-3)} + \frac{1}{2} e^{-j7(t-3)} \\
= \cos(4(t - 3)) + \cos(7(t - 3))
\]

For this simple example, one could find the output directly from the input-output equation

\[ x(t) \rightarrow y(t) = x(t - 3). \]
Consider a linear, time-invariant, continuous-time system with impulse response \( h(t) \) know that the response resulting from an input \( x(t) \) is

\[
y(i) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\,d\tau
\]

For complex exponential inputs of the form

\[
x(t) = \exp[j\omega t]
\]

the output of the system is

\[
y(t) = \int_{-\infty}^{\infty} h(\tau)\exp[j\omega (t - \tau)]\,d\tau
\]

\[
= \exp[j\omega t] \int_{-\infty}^{\infty} h(\tau)\exp[-j\omega \tau]\,d\tau
\]

By defining

\[
H(\omega) = \int_{-\infty}^{\infty} h(\tau)\exp[-j\omega \tau]\,d\tau
\]

we can write

\[
y(t) = H(\omega)\exp[j\omega t]
\]

To determine the response \( y(t) \) of an LTI system to a periodic input \( x(t) \) with the Fourier-series representation, we use the linearity property and \( y(t) = H(\omega)\exp[j\omega t] \) to obtain

\[
y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0)e^{jn\omega_0 t}
\]
Consider the circuit shown in Figure 3.6.2. The differential equation governing the system is

\[ i(t) = C \frac{dv(t)}{dt} + \frac{v(t)}{R} \]

For an input of the form \( i(t) = \exp[j\omega t] \), we expect the output \( v(t) \) to be \( v(t) = H(\omega) \exp[j\omega t] \). Substituting into the differential equation yields

\[ \exp[j\omega t] = Cj\omega H(\omega) \exp[j\omega t] + \frac{1}{R} H(\omega) \exp[j\omega t] \]

Canceling the \( \exp[j\omega t] \) term and solving for \( H(\omega) \), we have

\[ H(\omega) = \frac{1}{\frac{1}{R} + j\omega C} \]
Response of LTI systems to Complex Exponential: Example

Let us investigate the response of the system to a more complex input. Consider an input that is given by the periodic signal $x(t)$ in Example 3.3.1. The input signal is periodic with period 2 and $\omega_0 = \pi$, and we have found that (see slide# 13)

$$c_n = \begin{cases} 
\frac{2K}{jn\pi}, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}$$

From Equation (3.5.3), the output of the system in response to this periodic input is

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{2K}{jn\pi} \frac{1}{1/R + jn\pi C} \exp[jn\pi t]$$
Outline

- Introduction to frequency analysis
- Fourier series of CT periodic signals
- Signal Symmetry and CT Fourier Series
- Properties of CT Fourier series
- Convergence of the CT Fourier series
- Fourier Series of DT periodic signals
- Properties of DT Fourier series
- Response of LTI systems to periodic signals
- **Summary**
Fourier series: summary

- Sinusoid signals:
  - Can determine the period, frequency, magnitude and phase of a sinusoid signal from a given formula or plot

- Fourier series for periodic signals
  - Understand the meaning of Fourier series representation
  - Can calculate the Fourier series coefficients for simple signals (only require double sided)
  - Can sketch the line spectrum from the Fourier series coefficients
Fourier series: summary

**Fourier Series for Periodic Signals**

- Fourier series represents a periodic signal as a sum of many complex exponential signals, with frequencies being multiples of the fundamental frequency of the signal

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}, \quad a_k = \frac{1}{T/2} \int_{-T/2}^{T/2} x(t) e^{-j\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T}
\]

- Approximation accuracy increases with the number of terms used, Gibbs phenomenon

- Know special forms of forward and inverse transform for even and odd real functions
Fourier series: summary

- Steps for computing Fourier series:
  1. Identify period
  2. Write down equation for x(t)
  3. Observe if the signal has any symmetry (even or odd)
  4. Use the exponential equation (1) in previous slide, and if needed use Eq. (2) for the trigonometric coefficients
FS Summary: a quiz

Problem: Find the Fourier series coefficients of the periodic continuous signal: $x(t) = \cos\left(\frac{\pi}{3} t\right)$, $0 \leq t < 3$ and $T = 3$ is the period of the signal. Plot the spectrum (magnitude and phase) of $x(t)$. What is the spectrum of $x(t-4)$?

Solution: Using the definition of Fourier coefficients for a periodic continuous signal, the coefficients are:

$$a_k = \frac{1}{T} \int_0^T x(t)e^{-jk\omega_0 t} dt = \frac{1}{3} \int_0^3 \cos\left(\frac{\pi}{3} t\right)e^{-jk\frac{2\pi}{3} t} dt = \frac{1}{3} \int_0^3 e^{\frac{j\pi}{3} t} + e^{-\frac{j\pi}{3} t} e^{-jk\frac{2\pi}{3} t} dt$$

which would be simplified after some manipulations to:

$$\Rightarrow \quad a_k = \frac{4kj}{\pi(1 - 4k^2)}$$
Outline

- Introduction to frequency analysis
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- Properties of DT Fourier series
- Response of LTI systems to complex exponential
- Summary
- Appendix: Applications (not in the exam)
Applications of frequency-domain representation

- Clearly shows the frequency composition of a signal
- Can change the magnitude of any frequency component arbitrarily by a filtering operation
  - Lowpass -> smoothing, noise removal
  - Highpass -> edge/transition detection
  - High emphasis -> edge enhancement
- Can shift the central frequency by modulation
  - A core technique for communication, which uses modulation to multiplex many signals into a single composite signal, to be carried over the same physical medium
- Processing of speech and music signals
Typical Filters

- Lowpass -> smoothing, noise removal
- Highpass -> edge/transition detection
- Bandpass -> Retain only a certain frequency range
Low Pass Filtering
(Remove high freq, make signal smoother)

Filtering is done by a simple multiplication:

\[ Y(f) = X(f) \cdot H(f) \]

\( H(f) \) is designed to magnify or reduce the magnitude (and possibly change phase) of the original signal at different frequencies.

A pulse signal after low pass filtering (left) will have rounded corners.
High Pass Filtering
(remove low freq, detect edges)
Filtering in Temporal Domain (Convolution)

- Convolution theorem
  \[ X(f)H(f) \iff x(t) * h(t) \]
  \[ x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \]

- Interpretation of convolution operation
  - replacing each pixel by a weighted sum of its neighbors
  - Low-pass: the weights sum = weighted average
  - High-pass: the weighted sum = left neighbors \(-right neighbors\)
Communication: Signal Bandwidth

- Bandwidth of a signal is a critical feature when dealing with the transmission of this signal.
- A communication channel usually operates only at certain frequency range (called channel bandwidth).
  - The signal will be severely attenuated if it contains frequencies outside the range of the channel bandwidth.
  - To carry a signal in a channel, the signal needed to be modulated from its baseband to the channel bandwidth.
  - Multiple narrowband signals may be multiplexed to use a single wideband channel.
Signal bandwidth

- Highest frequency estimation in a signal:
  - Find the shortest interval between peak and valleys
Signal Bandwidth

- $f_{\text{min}}$ ($f_{\text{max}}$): lowest (highest) frequency where the FT magnitude is above a threshold
- Bandwidth:
  $$B = f_{\text{max}} - f_{\text{min}}$$
- The threshold is often chosen with respect to the peak magnitude, expressed in dB
  - $\text{dB}=10 \log_{10}(\text{ratio})$
  - 10 dB below peak = 1/10 of the peak value
  - 3 dB below = 1/2 of the peak value
Estimation of Maximum Frequency
Processing Speech & Music Signals

- Typical speech and music waveforms are semi-periodic
  - The fundamental period is called pitch period
  - The fundamental frequency ($f_0$)

- Spectral content
  - Within each short segment, a speech or music signal can be decomposed into a pure sinusoidal component with frequency $f_0$, and additional harmonic components with frequencies that are multiples of $f_0$.
  - The maximum frequency is usually several multiples of the fundamental frequency
  - Speech has a frequency span up to 4 KHz
  - Audio has a much wider spectrum, up to 22KHz
Sample Speech Waveform 1

Signal within each short time interval is periodic. The period $T$ is called "pitch". The pitch depends on the vowel being spoken, changes in time. $T \sim 70$ samples in this ex.

$f_0 = 1/T$ is the fundamental frequency (also known as formant frequency). $f_0 = 1/70fs = 315 \text{ Hz}$. $k\cdot f_0$ ($k$=integers) are the harmonic frequencies.
Numerical Calculation of CT Fourier Series

- The original signal is digitized, and then a Fast Fourier Transform (FFT) algorithm is applied, which yields samples of the FT at equally spaced intervals.
- For a signal that is very long, e.g. a speech signal or a music piece, spectrogram is used.
  - Fourier transforms over successive overlapping short intervals.
Sample Speech Spectrogram 1

```matlab
figure;
psd(x,256,fs);
```

Signal power drops sharply at about 4KHz

```matlab
figure;
specgram(x,256,fs);
```

Line spectra at multiple of \( f_0 \), maximum frequency about 4 KHz

What determines the maximum freq?
Sample Speech Waveform 2

Entire waveform

Blown-up of a section.

“In the course of a December tour in Yorkshire”
Speech Spectrogram 2

```matlab
figure;
psd(x,256,fs);
```

Signal power drops sharply at about 4KHz

```matlab
figure;
specgram(x,256,fs);
```

Line spectra at multiple of f0, maximum frequency about 4 KHz
Sample Music Waveform

Entire waveform

```matlab
[y, fs] = wavread('sc01_L.wav');
sound(y, fs);
figure; plot(y);
```

Blown-up of a section

```matlab
v = axis;
axis([1.1e4, 1.2e4, -.2, .2])
```

Music typically has more periodic structure than speech.
Structure depends on the note being played.
Sample Music Spectrogram

```
» figure; » psd(y,256,fs);
```

Signal power drops gradually in the entire frequency range

```
» figure; » specgram(y,256,fs);
```

Line spectra are more stationary, Frequencies above 4 KHz, more than 20KHz in this ex.