# Models of Blocking Probability in All-Optical Networks with and Without Wavelength Changers 

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#### Abstract

We introduce a traffic model for circuit-switched all-optical networks which we then use to calculate the blocking probability along a path for networks with and without wavelength changers. We investigate the effects of path length, switch size, and interference length (the expected number of hops shared by two sessions which share at least one hop) on blocking probability and the ability of wavelength changers to improve performance. Our model correctly predicts unobvious qualitative behavior demonstrated in simulations by other authors.


## I. Introduction

IN AN all-optical network, the signals remain in the optical domain from the origin to the destination. We focus on circuit-switched all-optical networks employing wavelengthdivision multiplexing (WDM) and switches which route signals based on their wavelength. The nodes of the network, which contain the frequency selective switches, are connected in an arbitrary mesh topology. The nodes may also contain wavelength changers.

We compare networks where every node contains wavelength changers to networks without any wavelength changers. Our main interest is the potential usefulness of wavelength changers in all-optical mesh networks, e.g., Level-2 of the ARPA-sponsored AT\&T/DEC/MIT AON [5]. Since these nodes have electrooptic counterparts, this study may also be helpful in comparing other architectures. For instance, a comparison of all-optical networks without wavelength changers to network with electrooptic nodes which are capable of routing any signal on any input port and wavelength to any output port and wavelength.

The particular situation we consider is shown in Fig. 1. Access station $A$ requests a session to station $B$ over some path of a mesh network and there are $H$ hops (fibers) from $A$ to $B$ on this path (we do not count the access or exit fibers). We consider networks where each session requires a full wavelength of bandwidth and there are $F$ available wavelengths.
For simplicity, we assume that $A$ and $B$ are not currently active at the time of the session request (for instance, each station may only contain one laser); however, our techniques can be easily generalized. Therefore, there are no busy wavelengths on the access or exit fiber and, in particular, a session cannot

[^0]enter the requested path at node $H+1$. However, sessions may enter or exit the path at each of the first $H$ intermediate nodes, provided that no two sessions on the same fiber use the same wavelength. Any session which uses at least one of the $H$ fibers on any wavelength is termed an interfering session.

With wavelength changers at every node, this is a conventional circuit-switched network. In this case, the request between $A$ and $B$ is blocked only if one of the $H$ fibers is full, (a fiber is full when it is supporting $F$ sessions on different wavelengths).

Without any wavelength changers, the session must use the same wavelength on each hop of the path. Therefore, a request can be honored on this path only if there exists a free wavelength, i.e., a wavelength which is unused on each of the $H$ fibers. Note that there is the possibility in such networks that requests will be blocked even if all links are supporting less than $F$ sessions. For instance, suppose that $H=F$ and wavelength $i$ is used on hop $i$ only. Then each fiber along this path has only one active session but there is no wavelength available to the request.

Obviously a network with wavelength changers is more flexible and has a smaller blocking probability. For this reason, there has recently been considerable research on the implementation of such devices, e.g., see [1]. However, quantitative results on the usefulness of wavelength changers have been mixed. The most wavelength-efficient topologies currently known do not require wavelength changers, and these topologies are nearly optimal in the sense that they use almost the minimum number of wavelengths [2], [3]. However, these networks have carefully designed topologies which are unlikely to be implemented on a national scale. On the other hand, simulations of random topologies have indicated a modest benefit of wavelength changers [4]. In the other extreme, examples can be constructed for which wavelength changers provide a very large performance gain [3].

We have developed an analytic traffic model which we use to approximate the blocking probability along a path with and without wavelength changers. The goals here are to identify important network parameters and to study the qualitative behavior of blocking probability as a function of these parameters.

Two other models have been concurrently proposed by Kovac̆evic [6] and Birman [7]. We note here that our model makes more simplistic traffic assumptions than either of the other two. However, our model is more analytically tractable, providing equations which we use to study the qualitative


Fig. 1. An $H$ hop request. Requested links are labeled. The other links are interfering links.
behavior of these networks. The other models both require recursive numerical calculation for even the simplest networks, thus making it difficult to draw general conclusions. In addition, our model applies to a wider variety of topologies than [6], as the latter model assumes that the state of successive links of the path are statistically independent. This assumption, which we temporarily make in Section II, correctly identifies path length as a key design parameter but tends to greatly overestimate the benefits of wavelength changers. Recently, Subramaniam has extended Kovačević's model to the case where only some nodes have wavelength changers [8]. Again the model requires recursive numerical evaluation.

The general form of our model is presented in Section IV. First, two interesting special cases are developed in Sections II and III.

The first form of the model, presented in Section II, is based on Lee's well known traffic model for circuit-switched networks [9]. Although simple, several interesting qualitative conclusions can be drawn about the effects of path length on blocking probability. We find that path length is a key design parameter for networks without wavelength changers. This conclusion has previously been pointed out by a variety of authors in different ways, e.g., by simulations in [4] and [10], by a theorem and an example in [3], and by the alternate blocking probability models in [6] and [7]. In order to keep the blocking probability small, path length must be kept small since it becomes less likely to find a free wavelength on all the hops of a path as the number of hops increases. That is, the number of interfering sessions on a path tends to increase with the number of hops.
In Section III, we extend the above model by using Pippenger's improvement [11] to Lee's model. We investigate the effects of path length and nodal degree (switch size) on blocking probability. We predict that the switch size is important because networks with large switches tend to mix sessions more than networks with small switches. That is, the number of interfering sessions on a path tends to increase with switch size as well as the number of hops. Although the effects are secondary to path length, we predict that they are significant in networks without wavelength changers if the switch size is small. In addition, we compare our results to simulations performed by Sivarajan and Ramaswami in [10].

A third important parameter, interference length, $L$, is a function of both the network topology and routing algorithm. Loosely speaking, the interference length is the number of hops shared by two sessions. The effects of $L$ are captured by the most general version of our model which is presented in Section IV. We argue that networks with large interference length reduce the need for wavelength changers since the number
of interfering sessions tends to decrease as the interference length increases.
Finally in Section V, we summarize our conclusions.

## II. The Effects of Path Length

We start by making the standard series independent link assumption introduced by Lee and commonly used in the analysis of circuit-switched networks [9], [12]. In particular, we assume that in steady state, a request sees a network where a wavelength is used on a hop statistically independently of other hops and other wavelengths. Lee's model tends to overestimate the blocking probability in circuit switched networks [12] and it would be surprising if that were not the case here.

Let $\rho$ be the probability that a wavelength is used on a hop. Note that since $\rho F$ is the expected number of busy wavelengths, $\rho$ is a measure of the fiber utilization along this path.
First, consider networks with wavelength changers. The probability $P_{b}^{\prime}$ that the session request between $A$ and $B$ is blocked is the probability that there exists a hop with all wavelengths used, i.e.,

$$
\begin{equation*}
P_{b}^{\prime}=1-\left(1-\rho^{F}\right)^{H} \tag{1}
\end{equation*}
$$

Let $q$ be the achievable utilization for a given blocking probability in networks with waveleng h changers, i.e.,

$$
\begin{equation*}
q \stackrel{\text { def }}{=}\left[1-\left(1-P_{b}^{\prime}\right)^{1 / H}\right]^{1 / F} \approx\left(\frac{P_{b}^{\prime}}{H}\right)^{1 / F} \tag{2}
\end{equation*}
$$

where the approximation is valid for small $P_{b}^{\prime} / H$.
In Fig. 2, we plot the achievable utilization $q$ for $P_{b}=10^{-3}$. The utilization is plotted as a function of the number of wavelengths for $H=5,10,20$ hops. Notice that the effect of path length on utilization is small. It is apparent that a good routing policy would minimize the congestion of links at the expense of more hops if possible, although minimum hop routing can be a good heuristic to minimize congestion by reducing the total bandwidth consumed by a single session. Notice also that $q$ rapidly approaches one as $F \rightarrow \infty$. This is a demonstration of the well known fact that large trunk groups are more efficient than small trunk groups.

Now consider a network without wavelength changers. Again let $\rho$ be the probability that a wavelength is used on a link. In the absence of wavelength changers, the probability of blocking $P_{b}$ is the probability that each wavelength is used on at least one of the $H$ hops, i.e.,

$$
\begin{equation*}
P_{b}=\left[1-(1-\rho)^{H}\right]^{F} . \tag{3}
\end{equation*}
$$



Fig. 2. Wavelength utilization increases with the number of wavelengths and the effects of $H$ are small in networks with wavelength changers. $P_{b}=10^{-3}$.


Fig. 3. Wavelength utilization increases with the number of wavelengths and the effects of $H$ are significant in networks without wavelength changers. $P_{b}=10^{-3}$.

Now let $p$ be the achievable utilization for a given blocking probability in networks without wavelength changers, i.e.,

$$
\begin{equation*}
p \stackrel{\text { def }}{=} 1-\left(1-P_{b}^{1 / F}\right)^{1 / H} \approx-\frac{1}{H} \ln \left(1-P_{b}^{1 / F}\right) \tag{4}
\end{equation*}
$$

where the approximation is valid for large $H$ and $P_{b}^{1 / F}$ not too close to one. To see this, notice that $(1-x)^{\alpha} \approx 1+\alpha \ln (1-x)$ as long as $\alpha \ln (1-x)$ is small. Note that the achievable utilization is inversely proportional to $H$.

In Fig. 3, we plot the achievable utilization $p$ for $P_{b}=10^{-3}$. The utilization is plotted as a function of the number of wavelengths for $H=5,10,20$ hops. Notice that unlike the previous case, the effect of path length is dramatic.

This would suggest that the diameter of a network without wavelength changers should be kept small, else fibers will be greatly underutilized. It would also suggest that a good routing policy for networks without wavelength changers consider path lengths in hops, as well as the congestion of links. ${ }^{1}$ Notice that the main reason for minimizing hops in networks without wavelength changers is different than the reason for minimizing hops in circuit switched networks or networks with wavelength changers. In the former case, we are trying to reduce the expected number of interfering sessions on a

[^1]path, whereas in the latter case, we are trying to minimize congestion on the links.

The model also predicts that $p$ increases with $F$, i.e., large trunk groups are more efficient than small trunk groups in networks without wavelength changers. Large trunk group efficiency was also observed analytically and using simulations by Ramaswami and Sivarajan [4]. The convergence appears to be so slow that this effect may be irrelevant for practical systems where the number of wavelengths is limited.

Notice also that our model predicts that $p$ approaches one as $F \rightarrow \infty$. Ramaswami and Sivarajan have shown that this prediction is incorrect for some networks in which $p$ is upper bounded by a value $<1$ [4]. The ramifications of this discrepancy are uncertain as the efficiency of "typical" mesh networks with a moderate number of wavelengths is still not clearly understood.

As a measure of the benefit of wavelength changers, define the gain $G=q / p$ as the increase in utilization for the same blocking probability. Setting $P_{b}=P_{b}^{\prime}$ and solving for $q / p$, we get

$$
\begin{align*}
G \stackrel{\text { def }}{=} \frac{q}{p} & =\frac{\left[1-\left(1-P_{b}\right)^{1 / H}\right]^{1 / F}}{1-\left(1-P_{b}^{1 / F}\right)^{1 / H}}  \tag{5}\\
& \approx H^{1-(1 / F)} \frac{P_{b}^{1 / F}}{-\ln \left(1-P_{b}^{1 / F}\right)} \tag{6}
\end{align*}
$$

for the wavelength changing gain. This gain comes at the cost of increased hardware. The approximation is valid for small $P_{b}$, large $H$, and moderate $F$ (so that $P_{b}^{1 / F}$ is not too close to one).

Typical plots of $G$ versus $F$ are shown in Fig. 4(a)-(c). In each figure, $G$ is shown as a function of $F$ for 5,10 , and 20 hops. Fig. 4(a) shows $G$ for a blocking probability $P_{b}=10^{-3}$. Likewise Fig. 4(b) and (c) shows $G$ for $P_{b}=10^{-4}$ and $P_{b}=10^{-5}$, respectively. The gain increases as the blocking probability decreases; however, the effect is small as long as $P_{b}$ is small.

Notice that $G=1$ if either $H=1$ or $F=1$ since in either of these cases there is no difference between a system with or without wavelength changers. So, for instance, wavelength changers are useless in two-stage (one hop) switching networks.

As $F$ increases, the gain increases until $G$ peaks somewhere near $F \approx 10(q \approx 0.5)$ for all cases shown. As can be seen from the figures, the maximum gain is close to $H / 2$. After peaking, the gain slowly decreases for the simple reason that large trunk groups are more efficient. The convergence is extremely slow since the convergence of $p$ is extremely slow. Our model also predicts that $G$ decreases to one as $F \rightarrow \infty$. However, for some networks, it has been shown that $G>1$ for all $F$ [3], [4].

It's interesting to note that even for a moderate number of wavelengths, we seem to be operating in a regime where there is diminishing returns for the use of wavelength changers. That is, as we increase the number of wavelengths, the node complexity increases and the benefit of the hardware decreases.

Now consider $G$ as a function of the number of hops $H$. Notice that for large $F$, the gain is roughly linear in the number


Fig. 4. Wavelength changing gain. The effects of $H$ are large.
of hops, basically because $q$ is nearly independent of $H$ and $p$ is inversely proportional to $H$. It can be shown that $G$ is never more than $G \leq H^{1-(1 / F)}$. Therefore, interestingly, for a two wavelength system, $G$ grows more slowly than $\sqrt{H}$.

In summary, for a moderate to large number of wavelengths, the benefits of wavelength changers increase with the number of hops and decrease with the number of wavelengths. The benefits also increase as the blocking probability decreases; however the effect is small as long as $P_{b}$ is small.

We argue in the next two sections that we have overestimated the gain in efficiency that wavelength changers provide.

## III. The Effectis of Switch Size

The model in the last section correctly identifies hop length as a major design criteria. However, that simplified approach does not identify another important parameter: switch size. As stated previously, large switches tend to mix signals more than small switches. In this section, we account for this effect. We argue that Lee's model overestimates the gain in efficiency that wavelength changers provide.


Fig. 5. Results of Sivarajan and Ramaswami [5].
We assume for simplicity that each node has $\Delta$ incoming and $\Delta$ outgoing unidirectional fibers, including the fibers on the path. In a node without wavelength changers, the signals on the input fibers are demultiplexed, each wavelength is switched independently, and then the signals are multiplexed on the appropriate output fiber, e.g., [5, Fig. 4]. In networks with wavelength changers, the nodes also contain wavelength changing devices before the demultiplexers and after the multiplexers. ${ }^{2}$

Define the $H$ links connecting $A$ to $B$ to be the requested links and the links entering or exiting the intermediate nodes the interfering links. Any call using a requested link is termed interfering. We assume that a wavelength is used on an incoming interfering link with probability $\rho$. Furthermore, we assume that all incoming interfering links are independent and that different wavelengths are independent. We also assume that the switches are equally likely to be in any of the $(\Delta!)^{F}$ possible states (each wavelength switch is equally likely to be in any of the $\Delta$ ! possible states where each state corresponds to a possible matching of the $\Delta$ inputs to the $\Delta$ outputs). For networks with changers, we assume that each changer is set statistically independently to one of $F$ states where each state corresponds to a permutation of the $F$ available wavelengths.

Notice that the probability that a wavelength $\lambda$ is used on an interfering link is not the probability that $\lambda$ is used on a requested link $i$. The former is by definition $\rho$. To calculate the latter probability $\rho_{i}$, notice that because the access link is assumed empty at the time of the request, $\lambda$ is not used on hop $i$ if the first $i$ switches connect hop zero to hop $i$ on $\lambda$. This occurs with probability $\Delta^{-i}$. If hop $i$ is not connected to hop zero on $\lambda$, then hop $i$ is connected to one of the interfering links on $\lambda$. In this case, $\lambda$ is used on hop $i$ with probability $\rho$. Therefore

$$
\begin{equation*}
\rho_{i}=\rho\left(1-\Delta^{-i}\right) . \tag{7}
\end{equation*}
$$

[^2]Notice that $\rho_{i}$ increases to $\rho$ fairly rapidly as we move down the chain. Also, the average utilization along the chain $H^{-1} \Sigma_{i=1}^{H} \rho_{i} \approx \rho$ if $\Delta^{H} \gg 1$. For these reasons, we will continue to call $\rho$ the utilization.

First, conider the blocking probability in a network without wavelength changers. Suppose that wavelength $\lambda$ is free on requested hop $i-1$. If the switch for $\lambda$ is set such that link $i-1$ is connected to link $i$, then $\lambda$ is not used on link $i$. The probability of this is $1 / \Delta$. Otherwise, link $i$ is fed by one of the interfering links on $\lambda$. In this case, $\lambda$ will not be used on $i$ with probability $(1-\rho)$. Therefore

$$
\begin{align*}
& \operatorname{Pr}\{\lambda \text { free on hop } \mid \lambda \text { free on hop } i-1\} \\
& \quad=\frac{1}{\Delta}+\left(1-\frac{1}{\Delta}\right)(1-\rho) \\
& \quad=1-\left(1-\frac{1}{\Delta}\right) \rho \tag{8}
\end{align*}
$$

Now since all wavelengths and all incoming interfering links are assumed to be independent, the blocking probability is easily calculated to be

$$
\begin{align*}
P_{b} & =[1-\operatorname{Pr}\{\lambda \text { free on all hops }\}]^{F} \\
& =\left[1-\prod_{i=1}^{H} \operatorname{Pr}\{\lambda \text { free on hop } i \mid \lambda \text { free on } i-1\}\right]^{F} \\
& =\left(1-\left[1-\left(1-\frac{1}{\Delta}\right) \rho\right]^{H}\right)^{F} \tag{9}
\end{align*}
$$

where hop zero is considered to be the fiber leaving station $A$ entering the first node, and we have assumed that station $A$ does not have any other active calls. Notice that we have also used the fact that the switches are set independently on each hop.

Notice that large $\Delta$ (more mixing) degrades performance. In one extreme $\Delta=1$, and there are no interfering links. In this case the $H$ hops look like one hop since all calls enter at node 1 and leave at node $H$. In the other extreme $\Delta \rightarrow \infty$, and the event $\lambda$ used on $i$ becomes independent of the event $\lambda$ used on $i-1$. This is so because with very high probability, the switch is set such that $i-1$ is not connected to $i$ on $\lambda .{ }^{3}$

Inverting (9) gives the achievable utilization for a network without wavelength changers. Denoting this utilization by $p$

$$
\begin{equation*}
p=\frac{\Delta}{\Delta-1}\left[1-\left(1-P_{b}^{1 / F}\right)^{1 / H}\right] \tag{10}
\end{equation*}
$$

Comparison with (3) shows that $p$ can be $\Delta /(\Delta-1)$ larger than predicted by the simplified model (the $\Delta=\infty$ model). We shall see shortly that the same conclusion does not hold for networks with wavelength changers, i.e., the effect of $\Delta$ is much smaller in this case. Therefore, finite $\Delta$ reduces the wavelength changing gain by about $(\Delta-1) / \Delta$. Since we observed earlier that $G \lesssim H / 2$ for many situations of interest, the benefits of using wavelength changers in networks with small diameter $D$ and small $\Delta$ are limited.

[^3]Before deriving the blocking probability for networks with wavelength changers, we compare our model to simulation results presented in [10]. We will see that our model makes unobvious qualitative predictions confirmed by those simulations. However, more simulations and experience are required for the model to be fully evaluated.

Sivarajan and Ramaswami considered DeBruijn graphs for store and forward as well as circuit-switched all-optical networks. We discuss their circuit-switched results here. They considered directed DeBruijn graphs where each node has inand out-degree $\Delta$ (a small number of nodes have self-loops). The number of nodes is $N=\Delta^{D}$ where $D$ is the diameter. The authors considered two possible designs for $N=1024$. The first network had $\Delta=4$ and $D=5$. The second had $\Delta=2$ and $D=10$. For each design, they simulated the DeBruijn network under the assumption that each node has $m=1$ and $m=5$ duplex session requests, i.e., if $A$ is talking to $B$ then $B$ is talking to $A$ and the session request is blocked if either direction of the request cannot be honored. The requests were handled sequentially, e.g., the first request cannot be blocked because the network is empty (in fact, the first $F$ requests cannot be blocked). Their results are shown in Fig. 5.

It is certainly not surprising that the ten diameter networks require more wavelengths than the five diameter networks under the same load [see (3)]. However, it is certainly not $a$ priori obvious that the five diameter network with five calls per node requires more wavelengths than the ten diameter network with one call per node.

Using (9), we calculate the steady state blocking probability as a function of the number of wavelengths. We set $\rho F \leq F$ to be the average number of sessions per link under the assumption that all calls are honored, i.e.,

$$
\begin{equation*}
\rho=\min \left\{1, \frac{m N \bar{H}}{N \Delta F}\right\}=\min \left\{1, \frac{m \bar{H}}{\Delta F}\right\} \tag{11}
\end{equation*}
$$

where $m N$ is the number of one-way session requests, ${ }^{4} \bar{H}$ is the average number of hops used by a call in each direction and $N \Delta$ is the total number of links in the network. ${ }^{5}$ For the blocking probability, we use (9) averaged over the number of hops, i.e.,

$$
P_{b} \approx \frac{2}{N} \sum_{H=1}^{D}(\Delta-1) \Delta^{H-1}\left(1-\left[1-\left(1-\frac{1}{\Delta}\right) \rho\right]^{H}\right)^{F}
$$

since a node can reach about $(\Delta-1) \Delta^{H-1}$ nodes in $H$ hops and there are $N=\Delta^{D}$ nodes. The factor of two in the numerator accounts for the fact that a duplex call request is blocked if either of the one way requests are blocked, ignoring the $o\left(P_{b}^{2}\right)$ term.

Our results are shown in Fig. 6. Notice that we identify the same ordering in terms of required number of wavelengths as the simulations; however, our blocking probabilities are larger. Likely reasons for this will be discussed momentarily.

[^4]

Fig. 6. Analytic approximation of $P_{b}$.


Fig. 7. Analytic approximation of $P_{b}$ using $\Delta=\infty$.

First, consider Fig. 7, which shows the blocking probability calculated using the independent requested link assumption (3). Notice that it predicts even higher blocking probabilities and incorrectly orders the designs. In particular, the five diameter with five calls per node (line $c$ ) requires less wavelengths than the ten diameter network with one call per node (line $b$ ), a conclusion not supported by the simulations. Our model suggests that the smaller diameter network requires more wavelengths because it has a higher load ( $m=5$ versus $m=1$ ) and because it has larger switches and therefore more mixing.

One reason our model predicts higher $P_{b}$ than the simulations is that we have calculated a steady state blocking probability, whereas in [10], the calls were set up sequentially and the network was not allowed to progress to steady state. Another likely reason our curves lie above the simulations is the wavelength assignment scheme used by Sivarajan and Ramaswami. In particular, if a path has more than one free wavelength available, the lowest numbered wavelength is used (first fit wavelength assignment). This has the effect of making the load on each wavelength different. We speculate that this reduces the blocking probability. Simulations on other topologies by [6] are consistent with this hypothesis.

Now consider the case with wavelength changers. It is trivial to show that the path blocking probability is [13]

$$
\begin{equation*}
P_{b}^{\prime}=1-\prod_{i=1}^{H}\left[1-\frac{\rho_{i}^{F}-\rho_{i-1}^{F}\left(\frac{1}{\Delta}+\rho\left(1-\frac{1}{\Delta}\right)\right)^{F}}{1-\rho_{i-1}^{F}}\right] \tag{12}
\end{equation*}
$$

where recall that

$$
\rho_{i}=\rho\left(1-\Delta^{-i}\right) \quad i=0,1,2, \cdots, H
$$

is the probability that a wavelength is used on hop $i$.
The effects of $\Delta$ on $P_{b}^{\prime}$ are small unless $\Delta, H$ and $F$ are small. For our purposes, a reasonable approximation for the achievable utilization $q$ is obtained by letting $\Delta \rightarrow \infty$. In that case, (12) becomes (1) and

$$
\begin{equation*}
q \approx\left(1-\left(1-P_{b}^{\prime}\right)^{1 / H}\right)^{1 / F} \tag{13}
\end{equation*}
$$

Even for $\Delta=2$, this approximation works well for large $H$ but slightly underestimates $q$ for paths of moderate length. For instance, for $\Delta=2$ and $F \geq 5$, the approximation is within $15 \%, 7 \%$, and $5 \%$ of the exact values for $H=5,10$, and 20 , respectively. For $F \leq 5$, the error is $20-25 \%$.

Now, define the gain to be the ratio of the average utilization along the path with wavelength changers versus without changers.

$$
\begin{align*}
G & =\frac{\frac{1}{H} \sum_{i=1}^{H}\left(1-\Delta^{-i}\right) q}{\frac{1}{H} \sum_{i=1}^{H}\left(1-\Delta^{-i}\right) p}=\frac{q}{p} \\
& \approx\left(\frac{\Delta-1}{\Delta}\right) \frac{\left[1-\left(1-P_{b}\right)^{1 / H}\right]^{1 / F}}{1-\left(1-P_{b}^{1 / F}\right)^{1 / H}} \tag{14}
\end{align*}
$$

which is $(\Delta-1) / \Delta$ smaller than (6).
In summary, for a moderate to large number of wavelengths, we predict that the benefit of wavelength changers increases with the number of hops and the nodal degree while decreasing with the number of wavelengths. Since the diameter $D$ of a network tends to decrease with increasing switch size, there is a topological design trade-off between $\Delta$ and $D$.

We expect the above model to accurately predict the qualitative behavior of regular networks such the DeBruijn Networks. We also expect the model to work fairly well for random graphs. ${ }^{6}$ However, as will be discussed in the next section, the model does not accurately predict the behavior of networks with special topologies.

[^5]
## IV. The Effects of Interference Length

In this section, we present the general form of the model. First, consider the following two motivating examples. Aggarwal et al. reported a network with nodal degree 2 and $O(N)$ hops, where $N$ is the number of stations [3]. This network had a lot of mixing; in particular, each call interfered with $O(N)$ other calls. The network required two wavelengths with wavelength changers and $N$ without changers. These results are consistent with the analysis of the previous sections, i.e., wavelength changers help a lot because of the large number of hops. Now consider a unidirectional ring network. The nodal degree is two, the average hop length is $O(N)$, and a call interferes with $O(N)$ other calls. However, this network requires $O(N)$ wavelengths with or without wavelength changers, i.e., wavelength changers can only reduce the required number of wavelengths by a constant factor for any $N$. This result is inconsistent with the previous model.

The important distinction between the two networks is that in [3], two sessions which share some link, share exactly one link. On the other hand, in a ring network, the interference length (the expected number of links shared by two sessions which share some link) is $O(N)$. We predict that as the interference length increases, the relative benefit of wavelength changers decreases. In the model of Section III, the assumption that each switch is set to an arbitrary state on each wavelength means that an interfering call stays on the path with probability $\Delta^{-1}$. Therefore, the interference length is implicitly assumed to be $\Delta /(\Delta-1)$, which is clearly an inappropriate assumption for a ring network.

We now propose a general traffic model appropriate for a wider variety of networks, including rings. Afterwards, we will calculate the blocking probability along an $H$ hop path with and without wavelength changers. Then we will be in a position to continue our discussion on the benefits of wavelength changers.

Consider Fig. 1 again. We model the traffic along this path at the time of a session request from $A$ to $B$. As before, we assume that there are no sessions on the access and/or exit fibers. At any node $i \geq 1$, sessions can leave or join the path. A session using hop $i$ which also uses hop $i-1$ is called a continuing session of hop $i$. A session using hop $i$ which has joined the path at node $i$ is called a session of hop $i$. The model is based on a pair of parameters: $P_{l}>0$, the probability a session leaves the path at a node, and $P_{n}>0$, the probability a new session joins the path at a node on an available wavelength. A wavelength is called available on hop $i$ if either it is unused on hop $i-1$ or the session using it on hop $i-1$ leaves. The model is formally defined by six assumptions. Note that below a session refers to an interfering session and $1 \leq i \leq H$.

1) All events on different wavelengths are statistically independent.
It is sufficient to describe the rest of the model for an arbitrary wavelength $\lambda$.
2) The probability that $\lambda$ is used on hop zero is $\rho_{0} \stackrel{\text { def }}{=} 0$.
3) Given the state of hop $i-1$, i.e., if $\lambda$ is used or not, the state of hop $i$ is statistically independent of the states of hops $0,1,2, \cdots, i-2$.
4) Given that $\lambda$ is used on hop $i-1$, we assume that the interfering session on $\lambda$ leaves the path at node $i$ with probability $P_{l}$. Else, it continues on the same wavelength. All sessions leave the path at node $\bar{H}+1$.
5) If $\lambda$ is not used on hop $i-1$, then a new session joins at node $i$ on $\lambda$ with probability $P_{n}$. No sessions join at node $H+1$.
6) If $\lambda$ is used on hop $i-1$ and the session using $\lambda$ on hop $i-1$ leaves at node $i$, then a new session joins at node $i$ on $\lambda$ with probability $P_{n}$. No calls join at node $H+1$.
Notice that we have assumed that interfering continuing sessions remain on the same wavelength. This is of course the case in networks without wavelength changers, but it might seem at first thought an inappropriate assumption otherwise. Clearly, in networks with wavelength changers, it is not necessary for a session to use the same wavelength on hop $i$ as on hop $i-1$. However, we argue that there is no loss in generality in assuming that the wavelengths of interfering continuing sessions are not changed (at the time of the call request). To see this, suppose $m$ sessions continue from hop $i-1$ to hop $i$ and that $n$ new calls enter at node $i, m+n \leq F$. Then with wavelength changers, we can without any loss in performance only change the wavelengths of the new sessions and leave the wavelengths of the continuing sessions unchanged. This may require rearranging and is possible to do for any path, but certainly not for all paths simultaneously.

The above discussion applies only to the interfering sessions at the time $A$ and $B$ make a request. Therefore, the blocking probabilities are still defined the same. In particular, in networks with wavelength changers the request is blocked only if a fiber is full. In networks without changers, the call is blocked if no wavelength is free on all the hops.

Notice that

$$
\begin{align*}
& \operatorname{Pr}(\lambda \text { used on } i \mid \lambda \text { not used on } i-1)=P_{n}  \tag{15}\\
& \operatorname{Pr}(\lambda \text { used on } i \mid \lambda \text { used on } i-1)=\left(1-P_{l}\right)+P_{l} P_{n} \tag{16}
\end{align*}
$$

where the first equation is the definition of $P_{n}$ and the second equation follows from the fact that if $\lambda$ is used on hop $i-1$ then it is used on hop $i$ if the call continues on the path or if it leaves and a new call arrives. Notice that $P_{l}=1$ implies that a wavelength is used on successive links independently. This corresponds to the model of Section II. This assumption was also made in [6]. Also, if $P_{n}=q$, and $P_{l}=1-1 / \Delta$, then $1-P_{l}+P_{l} P_{n}=1 / \Delta+q(1-1 / \Delta)$, and we get the model of Section III. We will further discuss these relationships later.

From assumptions 1) and 2), the number of calls on hop $i$ is binomial distributed. The probability $\rho_{i}$ that a wavelength $\lambda$ is used on hop $i$ is easily calculated; if $\lambda$ is used on hop $i-1$, it is used on hop $i$ with probability $1-P_{l}+P_{l} P_{n}$, otherwise it is used with probability $P_{n}$. Therefore, $\rho_{i}=$ $\left(1-P_{l}+P_{l} P_{n}\right) \rho_{i-1}+P_{n}\left(1-\rho_{i-1}\right)$ and

$$
\begin{equation*}
\rho_{i}=\rho\left\{1-\left[1-\left(P_{l}+P_{n}-P_{l} P_{n}\right)\right]^{i}\right\} \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho \stackrel{\text { def }}{=} \frac{P_{n}}{P_{n}+P_{l}-P_{n} P_{l}} . \tag{18}
\end{equation*}
$$

Note that $P_{l}+P_{n}-P_{l} P_{n}$ is the probability that at a node, a new call joins or an old call leaves. If a new call joins or an old call leaves then the node is either adding or removing a call, i.e., there is mixing. We will see that the benefit of wavelength changers increases with this probability since increasing $P_{l}$ or $P_{n}$ increases the average number of interfering calls (increasing $P_{l}$ increases the expected number of available wavelengths at a node, and therefore increases the expected number of new calls).
Since $P_{l}+P_{n}-P_{l} P_{n}>0, \rho_{i}$ increases with $i$ toward $\rho$. Furthermore, if this probability is large, the convergence is rapid and the average utilization along the path is $\approx \rho$. For these reasons, we continue to call $\rho$ the utilization.
Before using (15)-(17) to derive the blocking probability with and without changers, we discuss how $P_{n}$ and $P_{l}$ might be determined when the interference length $L$ and the traffic between stations is known. Suppose a session joins the path on $\lambda$ at node $i$. Then with probability $P_{l}$, the session leaves the path at node $i+1$. The expected number of hops used by this session is

$$
\begin{equation*}
\sum_{h=i}^{H}(h-i+1)\left(1-P_{l}\right)^{h-i} P_{l} \approx \frac{1}{P_{l}} \tag{19}
\end{equation*}
$$

where the approximation ignores the truncation effects caused by the finite length of the path and is valid as long as $i+L \ll H$. We will assume this approximation is valid for the rest of this paper and set $P_{l}=1 / L$. Now from (18)

$$
\begin{equation*}
P_{n}=\frac{\rho P_{l}}{1-\rho\left(1-P_{l}\right)}=\frac{\rho}{L-\rho(L-1)} \tag{20}
\end{equation*}
$$

so $P_{n}$ can be determined from $L$ and $\rho$. A reasonable estimate for $\rho$ is

$$
\begin{equation*}
\rho=\frac{N \gamma \bar{H}}{F f} \tag{21}
\end{equation*}
$$

where $N$ is the number of stations, $\gamma$ is the expected number of active outgoing calls per station, $\bar{H}$ is the expected number of hops/call, $F$ is the number of wavelengths, and $f$ is the total number of fibers in the network.

To illustrate our approach, consider an $N$ node bidirectional ring network with a load of $\gamma$ Erlangs per node. The average interference length for a call request traveling half-way around the ring is about $N / 4$; therefore, we set $P_{l}=4 / N$. The utilization is $\rho=\gamma N / 8 F$. Therefore, $P_{n} \approx 0.5 \gamma / F-$ $0.125 \gamma N$. Note that $125 \gamma N$ is the expected load per link, so that even in networks with wavelength changers, we would expect $F \gg 0.125 \gamma N$. We will use these numbers below to estimate the gain in utilization that wavelength changers can provide.
It is a simple matter to calculate the blocking probability without wavelength changers, i.e.,

$$
\begin{align*}
P_{b} & =\left[1-\prod_{i=1}^{H} \operatorname{Pr}\{\lambda \text { free on } i \mid \lambda \text { free on } 0,1, \cdots, i-1\}\right]^{F} \\
& =\left[1-\left(1-P_{n}\right)^{H}\right]^{F} . \tag{22}
\end{align*}
$$

As can be seen, the path blocking probability in a network without wavelength changers is directly dependent on the
probability a new call joins the path $P_{n}$, and only indirectly dependent on the utilization $\rho$ [through (18)]. Therefore, if the interference length is large, it is possible to have a very large utilization $\rho \approx 1$ and still have a very small blocking probability. To see this more clearly, invert (22) for $P_{n}$ and use (18). Then the achievable utilization for a given blocking probability is

$$
\begin{equation*}
p=\frac{P_{n}}{P_{n}+P_{l}-P_{n} P_{l}} \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{n}=1-\left(1-P_{b}^{1 / F}\right)^{/ / H} \tag{24}
\end{equation*}
$$

We see that when calls tend to stay together (small $P_{l}$ ), the utilization can approach one.

With wavelength changers, the blocking probability is [13]

$$
\begin{equation*}
P_{b}^{\prime}=1-\prod_{i=1}^{H}\left[1-\frac{\rho_{i}^{F}-\left(1-P_{l}+P_{l} P_{n}\right)^{F} \rho_{i-1}^{F}}{1-\rho_{i-1}^{F}}\right] \tag{25}
\end{equation*}
$$

Bounds on $P_{b}^{\prime}$ can be obtained as follows. $P_{b}^{\prime}$ is at least the probability that hop $H$ is full, i.e., $P_{b}^{\prime} \geq \rho_{H}^{F}$. Also, $P_{b}^{\prime}$ is the probability that some link is full, which is no more than $\Sigma_{i=1}^{H} \rho_{i}^{F}$. The sum is no more than $H \rho_{H}^{F} \leq H \rho^{F}$ since the $\rho_{i}$ 's are increasing to $\rho$. Therefore

$$
\rho\left\{1-\left[1-\left(P_{n}+P_{l}-P_{n} P_{l}\right)\right]^{H}\right\} \leq\left(P_{b}^{\prime}\right)^{1 / F} \leq H^{1 / F} \rho .
$$

Now a good approximation for $\rho$ can be obtained if $H^{1 / F} \approx$ 1 and if $H>2 L$. First, since $1-\left(P_{n}+P_{l}-P_{n} P_{l}\right) \leq 1-P_{l}=$ $1-1 / L$

$$
\begin{equation*}
\rho\left[1-\left(1-\frac{1}{L}\right)^{H}\right] \leq\left(P_{b}^{\prime}\right)^{1 / F} \leq H^{1 / F} \rho \tag{26}
\end{equation*}
$$

Now if $H>2 L,(1-1 / L)^{H} \leq e^{-2}=0.13$. So for most cases of interest, the achievable utilization in a network with wavelength changers $q$ is approximately $\left(P_{b}^{\prime} / H\right)^{1 / F}$, which is also approximately

$$
\begin{equation*}
q \approx\left[1-\left(1-P_{b}^{\prime}\right)^{1 / H}\right]^{1 / F} . \tag{27}
\end{equation*}
$$

Even if $L=H / 2$, the error is only about $13 \%$ for large $F$.
Equations (22) and (25) can be used to estimate blocking probabilities when $P_{l}$ and $P_{n}$ are known. Our goal here is to estimate the performance gain of wavelength changers for the same path and routing algorithm (same $L$ ). We use $G=q / p$, where, as before, $q$ and $p$ are the achievable utilizations for networks with and without wavelength changers for the same blocking probability, same number of wavelengths, and same interference length $L .{ }^{7}$ From (24) and (27) with $P_{b}^{\prime}=P_{b}$, we get after simplification

$$
\begin{align*}
G \approx & \frac{\left[1-\left(1-P_{b}\right)^{1 / H}\right]^{1 / F}}{1-\left(1-P_{b}^{1 / F}\right)^{1 / H}} \\
& \times\left\{\left[1-\left(1-P_{b}^{1 / F}\right)^{1 / H}\right]\left(1-\frac{1}{L}\right)+\frac{1}{L}\right\} \tag{28}
\end{align*}
$$

[^6]

Fig. 8. Effect of interference length on the gain.

Equation (28) compares a system with wavelength changers to a system without wavelength changers for the same blocking probability and same interference length. Since the two systems have different utilizations, the new call probability $P_{n}$ will be different in the two systems. Let $P_{n}=1-(1-$ $\left.P_{b}^{1 / F}\right)^{1 / H}$ be the new call probability for a system without wavelength changers. Then $G$ can be expressed as

$$
\begin{align*}
G & \approx \frac{\left[1-\left(1-P_{b}\right)^{1 / H}\right]^{1 / F}}{1-\left(1-P_{b}^{1 / F}\right)^{1 / H}}\left(P_{n}+P_{l}-P_{n} P_{l}\right) \\
& =G_{o}\left(P_{n}+P_{l}-P_{n} P_{l}\right) \tag{29}
\end{align*}
$$

where $G_{o}$ is the gain when the interference length $L=1$, or equivalently the gain when successive links are assumed independent. The gain increases with Pr (new call joins or old call leaves) since increasing this probability increases mixing. A plot of $G$ is shown in Fig. 8 for a 20 -hop path, a blocking probability of $10^{-3}$, and interference lengths of $L=1,2,4$. As can be seen, the gain is proportional to $H / L$. Notice the similarity between these curves and the curves for $H=20,10,5$ shown in Fig. 4(a). In terms of the gain, an $H$ hop path with interference length $L$ looks like an $H / L$ hop path.

For example, we earlier calculated $P_{l}=4 / N$ and $P_{n} \approx$ $0.5 \gamma /(F-0.125 \gamma N)$ for a request traveling halfway around an $N$ node ring network with load $\gamma$ Erlangs per node. For these values, the path length $H=N / 2$ but the effective path length $H / L=2$. Equation (27) predicts gains of roughly $40-60 \%$ for $F \geq 10$ and various values of $N$. These results are consistent with simulations in [6].

Intuitively, adding wavelength changers to a network can only significantly increase fiber utilization if bandwidth is being underutilized on a link (since the gain is the increase in utilization) and if calls have lots of interferers (else many wavelengths would be free on the path and the utilization could be increased without changers). However if calls have lots of interferers and if interfering calls tend to stay together, it is impossible to greatly underutilize a link.

## V. CONClUSION

We modeled the probability of a path being blocked with and without wavelength changers under simple traffic models. We
then used this model to identify important network parameters and study the qualitative behavior of blocking probability as a function of these parameters.
The blocking probability with and without wavelength changers increase with the number of hops $H$. However, the effect is much more dramatic in networks without wavelength changers since the number of calls a given call shares some link with tends to increase with $H$. That is, networks with large diameter $D$ tend to have a lot of mixing. It therefore becomes harder to find a wavelength which is not used by any interfering call. This has lead researchers to conclude that minimizing the network diameter and employing minimum hop routings are reasonable heuristics for networks without wavelength changers, e.g., [14]. We concur with two caveats.
First, in regular networks such as the DeBruijn Networks, the amount of call mixing tends to increase with the switch size $\Delta$. That is, the number of interfering calls tends to increase with $\Delta$ as well as $H$. We therefore expect $P_{b}$ to increase with $\Delta$ and $H$. Since the diameter of a network tends to decrease with increasing switch size, there is a topological design trade-off between $\Delta$ and $D$.

Second, networks with large interference length $L$ have smaller blocking probability than networks with small interference length. In particular, the effective path length $H / L$ is the most important parameter in our model. We estimated a gain in fiber utilization using wavelength changers of no more than about $H / 2 L$. Therefore, we might choose to design networks with larger diameter if this permitted us to increase the interference length. Also, for a given topology, we may choose not to do minimum hop routing if this allowed us to decrease $H / L$.

Finally, we note that given the simple traffic assumptions we make here, the numerical accuracy of our model is questionable in trying to predict the behavior of simulations, e.g., Poisson traffic, exponential holding times. However, if the interference length is large, our model is likely to be more accurate than any model which makes the independent link assumption, e.g., [6]. In addition, we have recently extended this model with the goal of numerical accuracy. Initial results have been very accurate for small mesh networks with fewer than 30 wavelengths [15].

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[^1]:    ${ }^{1}$ However, we will argue in the next two sections that this model overestimates this effect.

[^2]:    ${ }^{2}$ This design is rearrangeably nonblocking. Strict sense nonblocking switches can be designed, e.g., [5, Fig. 5], but their introduction would only complicate the discussion here.

[^3]:    ${ }^{3}$ If the carried load between $A$ and $B$ is more than one call, then this assumption is violated. In this case, the effect of mixing is diminished since there are more calls from $A$ to $B$ and less interfering calls.

[^4]:    ${ }^{4}$ In [13], we incorrectly used $2 m N$ as the number of one-way session requests. This error effects Figs. 6 and 7, but does not change any of the conclusions reached in [13].
    ${ }^{5} \mathrm{~A}$ better model for large $P_{b}$ would be to set $\rho=m \bar{H}\left(1-P_{b}\right) / \Delta F$.

[^5]:    ${ }^{6}$ For instance, in [4], the authors simulated a wavelength changing gain (measured in a related but slightly different way than ours) for 1024 and 16 node random networks with $\Delta=4, F=10$, and blocking probability 01. They predicted a utilization gain of 1.4 for the 1024 network and measured no gain in efficiency for the 16 -node network. Using $H \approx \log _{\Delta} N$, our models predict a maximum gain of 2.2 for $N=1024$ and 1.1 for $N=16$.

[^6]:    ${ }^{7}$ This definition is consistent with the previous two sections where the interference length was one and $\Delta /(\Delta-1)$.

