Chapter 2

The Z-transform

OBJECTIVE: To see how Z-transforms can be used to answer questions concerning **stability, causality** and **convergence**.
To see how Z-transforms can be used to determine system output.
To study the use of Z-transforms in the systems design process.
To see how Z-transforms are useful in the analysis of frequency response.

PRE LAB - Signal Acquisition

Prior to coming to the lab you should acquire a real-life signal of some sort. R ecall that a signal is a *stream* of data. The data in your stream may be real or complex. The stream should be single channel and one-dimesional. It should of course be a digital signal.

Possible ways to acquire a signal:

- Run an experiment at work/home. Capture the results of the experiment on t he computer.
- Run a software experiment on the computer.
- Examine sources on the computers at Concordia (images, sounds).
- Get signals from the Internet. (Explore WWW)
- Any way you want.

Look at the last Problem in this experiment to see what information you should b e sure to capture. Mainly you will want the sampling rate.

During the lab period you must: Place at least 1000 samples of your signal in the directory designated by the TA and hand in a write-up to the LAB TA that:

- describes the signal that your acquired, how you acquired it

- describes why it is an interesting/important signal. (This will likely be the most interes ting part so don't be afraid to indulge in a very good description).
- State the sampling rate and the unit of the samples (for example if the signal is the output of a stereo amplifier to a speaker the units might be volts.) If you can't say what unit it is, then explain it as best you understand it.

(The file containing your signal should be ASCII readable format. It should be s uch that it can be loaded into MATLAB using the load command. This means that the file ca n take one of two formats

- The number representing each sample is placed on a new line. Thus there is only one number on each line.
- The numbers representing the samples are all on the same line each separat ed by one or more spaces. This file contains only one line.

2.1 Z transform

Laplace transform can be considered to be a generalization of the Fourier Transform (FT) in the continuous-time domain. Similarly, in the discrete-time domain, the Z-transforms is a generalization of the *discrete-time Fourier Transform* (DTFT). The Z-transforms plays an important role in both analysis and design of discrete-time systems. It provides another domain in which signal and systems can be examined.

The two-sided or bilateral Z-transforms can be written as a summation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

or notationally as $X(z) = \mathcal{Z}\{x(n)\}$. Two important properties of the Z-transforms are apparent from its definition. First, the transform is linear and its second property enables the treatment of shifted sequences, i.e,

$$x(n) \longleftrightarrow X(z) \quad \& \quad x(n-k) \longleftrightarrow z^{-k} X(z)$$

Z-transforms has many other valuable properties that make it a powerful analysis tool. One of the most important property is the equivalence of convolution of two sequences and the multiplication of their Z-transforms in the transform domain, i.e,

$$x(n) \otimes y(n) \longleftrightarrow X(z) Y(z)$$

where $\otimes$ denotes the convolution operation.

By expressing the complex variable z in the polar form as $z = re^{j\omega}$, Z-transforms has an interpretation in terms of Fourier Transform. With z, so expressed, the Z-transforms becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\omega n}$$

If $r = 1$, i.e, $\mid z \mid = 1$, the Z-transforms is equivalent to the DTFT of the sequence $x(n)$.

In this experiment, we motivate the use of Z-transforms by the role it plays in both design and analysis of discrete-time systems.

Problem 1

The focus in this problem is on demonstrating how the Z-transform answers questions concerning **causality, stability** and **convergence**. Consider the second-order system transfer function

$$H(z) = \frac{1 - 1.7z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

and an input sequence specified by

$$X(z) = \frac{1}{1 - .9z^{-1}}.$$

The input sequence is taken to have a *region of convergence* (`ROC`) given by $|z| > .9$.

1. Derive theoretically the inverse z-transform of $H(z)$ associated with the different regions of convergence. Indicate all relevant steps. On a polar plot indicate the pole and zero locations of the second-order system, as well as the corresponding ROCs. Indicate on the polar plot where the sampling rate is located. Discuss the causality and stability of each `ROC`. Does the Fourier Transform $H(z = \mathrm{e}^{j\omega})$ exist for each `ROC`?

2. Theoretically calculate the convolution of the input sequence corresponding to $X(z)$ with the stable sequence(s) corresponding to $H(z)$. Show all relevant analytical calculations. Does the Fourier Transform exist for all `ROC` of $H(z)X(z)$?

3. Generate the convolution of the input sequence for X(z) with the stable sequence(s) corresponding to $H(z)$ using the Matlab routine **conv**. Consider an input sequence and impulse response of length $N = 10$ samples and length $N = 250$ samples. These results are to be compared with the analytical expression you derived in **Part 2**. From the analytical expression(s) of **Part 2** generate two sequences of lengths $N = 19$ and $N = 499$ samples. Plot your results using the **comb** routine. Label each convolution sequence accordingly. Do the numerical convolution results agree with the theoretical expression? Why or why not?

Problem 2

Knowledge of the impulse response of a system is important in digital signal processing. In this problem, you will find the system transfer function by determining the system impulse response of an unknown system, $T[\cdot]$.

1. Find the impulse response of the unknown system defined by the function **func_8**. This can be done by using an impulse as the input to **func_8**. Use **stem** to show the results.

   (a) Generate an N-point sample input

   $$x(n) = \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & 0 < n < N - 1 \end{cases}$$

   Use $N = 32$ for this experiment.

   (b) Determine the output of the unknown system defined by func_8. Type **help func_8** for its usage.

2. (a) Define $x_1(n) = 2\delta(n-3)$ using 100 samples as in (1).

   (b) Find the output $y_1(n) = T[x_1(n)]$.

   (c) Using the convolution routine **conv** in matlab, evaluate $y_2(n)$ as

   $$y_2(n) = x_1(n) \otimes h(n)$$

3. Is the system $T[\cdot]$ LTI? Is the system stable? Explain.

4. If $T[\cdot]$ is LTI and given that its impulse response is exponential, find the transfer function. Why must the system be LTI, so that its transfer function can be determined? Is it necessary that it is also stable?

Recall that the frequency response of a system is equivalent to evaluating the transfer function, $H(z)$, of a system on the unit circle, i.e, let $z = e^{j\omega}$,

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

Using this methodology, it is possible to sketch the frequency response of the system that is related to its poles and zeros. Here we will only consider the magnitude response, while phase response can be considered in a similar way and is left as an exercise. Let $H(z)$ be written in its factorized form

$$H(z) = \frac{C(z-b_1)(z-b_2)\cdots(z-b_M)}{(z-a_1)(z-a_2)\cdots(z-a_N)}$$

then the magnitude response can be written as

$$| H(e^{j\omega}) |= \frac{| c || (e^{j\omega} - b_1) || (e^{j\omega} - b_2) | \cdots | (e^{j\omega} - b_N) |}{| (e^{j\omega} - a_1) || (e^{j\omega} - a_2) | \cdots | (e^{j\omega} - a_M) |}$$

A geometrical interpretation can be formed by evaluating the magnitude response of an arbitrary term $(e^{j\omega} - z_i)$. The point $e^{j\omega}$ lies on the unit circle at an angle of ω as in Fig. 3.1. The location of $z_i =| z_i | e^{j\theta}$ is also shown in the figure. The quantity of interest is the magnitude of the difference between these two vectors as shown in Fig. 3.1.

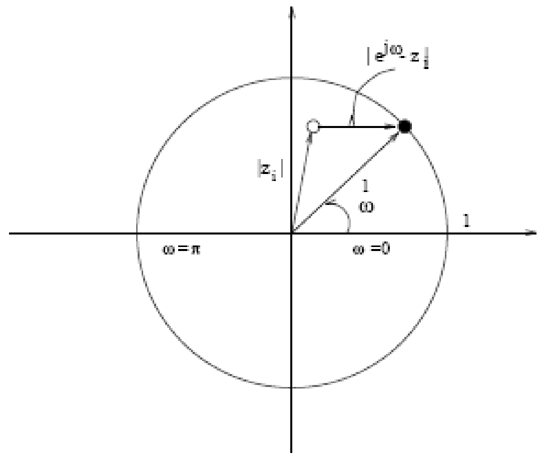

Figure 2.1: Geometric interpretation of the contribution of the magnitude response due to a single root at an arbitrary frequency

It is evident, that as the vector $e^{j\omega}$ rotates from 0 to 2π in a counter-clock-wise direction, the magnitude of the difference vector $(e^{j\omega} - z_i)$ also varies. The implication of the geometric representation is that we can evaluate the magnitude response due to the root z_i, as the length of this difference vector at different frequencies in the range $[0, 2\pi]$.

In Fig. 3.1, at $\omega = 0$, the magnitude of the difference vector is some finite value. As ω starts to increase this value starts decreasing and is minimum when $\omega = \theta$, the angle of the root z_i. If we continue to increase ω, the difference magnitude again starts to increase until $\omega = 2\pi$, at which point everything is repeated again.

From this geometrical interpretation, the idea of periodicity of DTFTs is evident, That is, if we continue to rotate $e^{j\omega}$ around the unit circle beyond 2π, the magnitude $(| e^{j\omega} - z_i |)$ starts to repeat. Each such revolution of the around the unit circle corresponds to a single period which leads to the idea of periodicity of the DTFTs.

The example of Fig. 3.1, illustrates how the graphical representation can be generalized to represent any arbitrary system function. The magnitude response can be represented as the product of individual difference vectors for the zeros divided by the product of difference vectors for the poles at each frequency.

At this time, several useful observations relating the magnitude response and pole/zero locations can be made. Zeros close to the unit circle cause the DTFT magnitude to dip in the region near the zeros and the minimum value of the dip occurs at the angle of the zero. If the zero is on the unit circle, the DTFT magnitude is zero for that frequency (angle of zero). Similarly, the poles close to the unit circle force a peak in the magnitude response at frequencies close to the angle of the poles. With these observations in mind, we are able to generate a rule for sketching the magnitude response of $H(z)$. As we move around the unit circle the magnitude peaks when we pass close to a pole and dips when we pass near a zero. The sharpness of the peaks and valleys depend on the closeness of the poles and zeros on the unit circle. Poles and zeros far from the unit circle do not affect the magnitude response significantly.

Problem 3

1. Given an all-pole filter (IIR) transfer function

$$H(z) = \frac{1}{1 - .5z^{-1} + .2z^{-2} - .1z^{-3} + .007z^{-4} + .14z^{-5} + .15z^{-6}}$$

   (a) Find its magnitude response using the matlab routine **freqz**. From the magnitude response what can be said about the pole locations?

   (b) From the magnitude response try to sketch the pole/zero plot.

   (c) Using **polezero** routine plot the true pole/zero locations.

2. Repeat (1) for the all-zero (FIR) filter transfer function given by

$$H(z) = 1 + 1.9z^{-1} + .8z^{-2} - .8z^{-3} - .7z^{-4}$$

3. Repeat (1) for the following transfer function (use a value of $\alpha = 0.8$)

$$H(z) = \frac{1 - 2cos\omega_o z^{-1} + z^{-2}}{1 - 2\alpha cos\omega_o z^{-1}\alpha^2 + z^{-2}}, \qquad \omega_o = \pi/4$$

- What visible clues in the magnitude response of the given transfer function provide information about the pole/zero locations?
- How are these clues affected by the poles and zeros away from the unit circle?
- How does α affect the pole locations in (3) above? What values can α assume such that it is stable function?
- Based on your sketch of the magnitude response from the pole/zero plot, what can you say about the function of this filter (What does it do?) in (3) above?

Problem 4

In this discussion the effect of pole locations on the output of a discrete-time system is considered. The effect poles have on system dampening behavior is investigated by examining a simple system.

1. Using Matlab, implement and test a **two pole oscillator**. An oscillator can be understood as a system that continues to generate an output between prescribed limits, once it has been started, i.e, marginally stable system. The sampling rate is 1200`Hz`, with the resonant or center frequency of the system taken to be 350`Hz`. The output of the oscillator is to be a sequence of **real numbers**. A two pole oscillator has complex-conjugate pair of poles, that ensure a **real** system. The output of a two pole oscillator is a sinusoid of frequency 350 Hz. Design notes should include a polar plot of the pole locations with sampling rate and resonant frequency appropriately indicated on the plot. The difference equation of the oscillator should be included with the initial conditions required to initiate oscillation. Can any input to the oscillator be used to start the system oscillating with a bounded amplitude? What if a unit step of finite amplitude is used as the initial excitation? Would the output of the oscillator have a bounded output? Generate and plot 32 samples of the oscillator output. The time axis should be properly scaled in seconds. **Hint**: Examine the input-output relationship in the z-domain.

2. The output of the oscillator is to be dampened. Explain what must be done in order to dampen the output. Note that dampening and attenuation are different. Choose an example where the output of the system is dampened and plot the output of the system for 256 samples.