Chapter 5

FIR Filter Design: Windowing

OBJECTIVE: To study the effects of windowing a data sequence.
To use windows for FIR filter design.
To Approximate IIR filters with FIR using windows.

Windowing - Resolution and Leakage

In practice the discrete-time system can only operate on a finite segment of data at any one point in time. In this section, we discuss the effect segmentation has on the `DFT` spectrum with respect to spectral resolution and leakage. To develop the concepts of resolution and leakage we begin by first considering data windowing.

Denote $x[n]$ as a data sequence of arbitrary length. Letting

$$w[n] = \begin{cases} w_n & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

define a window sequence of length N. We now denote the windowing of $x[n]$ by $w[n]$ as

$$x_w[n] = \begin{cases} x[n]w[n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

where $\{x_w[n]\}$ denotes the windowed data sequence.

To discuss some of the issues which arise because of windowing consider the example of a rectangular window. Let

$$w[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} .$$

Taking the z-transform of the sequence $\{w[n]\}$,

$$W(z) = T_s \sum_{n=-\infty}^{\infty} w[n] z^{-n}$$

the DTFT of the window sequence can be expressed as

$$W(z = \mathsf{e}^{j\omega}) = W(\mathsf{e}^{j\omega}) = T_s \mathsf{e}^{-j0.5\omega T_s(N-1)} \frac{\mathtt{sin}(0.5\omega T_s N)}{\mathtt{sin}(0.5\omega T_s)}$$

where Fig. 5.1a shows the magnitude of $W(\mathsf{e}^{j\omega})$ plotted over the normalized frequency interval $0 \leq \omega < 2\pi$. Now, consider the sampled sinusoid described by

$$x[n] = \alpha \mathtt{cos}(\omega_c(nT_s - \tau))$$

where α denotes amplitude and τ delay. The DTFT of $x[n]$, denoted by $X(\mathsf{e}^{j\omega})$, is shown in Fig. 5.1b. Windowing $\{x[n]\}$ by the sequence $\{w[n]\}$ yields the windowed sequence given by

$$x_w[n] = \begin{cases} x[n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} .$$

The resulting DTFT of the windowed sequence can be expressed as in terms of the convolution of $X(e^{j\omega})$ and $W(\mathsf{e}^{j\omega})$,

$$X_w(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\theta-\omega)}) d\theta$$

as shown in Fig. 5.1c. As indicated by Fig. 5.1c the effect of the convolution has been to broaden the sharp impulsive spectrum of $X(\mathsf{e}^{j\omega})$. Examining the expression for $W(\mathsf{e}^{j\omega})$, the width of the mainlobe (see Fig. 5.1a) is centered about

$$\omega_c + \frac{2\pi m}{T_s} \ \mathtt{rad} \backslash \mathtt{sec} \quad \text{for } m = \ldots, -2, -1, 0, 1, 2, \ldots$$

We note that the mainlobe width is inversely proportional to the length of the window. In other words as N increases the transform $W(\mathsf{e}^{j\omega})$ approaches the delta function. As N increases, what effect would this have on the spectrum of the windowed sequence $x_w[n]$?

Having reached this point in the discussion the issue of spectral resolution, a fundamental limitation of the DFT, can be addressed. Qualitatively resolution indicates the degree to which detail in the spectrum can be ascertained. For instance consider the sum of two sampled sinusoids that are closely spaced in frequency. The effect of taking an N-point DFT of this data sequence will tend to smear the spectrum of the sinusoids due to the convolution of $W(\mathsf{e}^{j\omega})$ with the DTFT of the sampled signal $X(e^{j\omega})$. Note that by taking only N points of the sequence we have effectively used a rectangular window on the data. If the two sinusoids are too close, this effect of smearing will result in a spectrum which only indicates the presence of one sinusoid, at this point the limit of resolution has been reached for a given N and the type of window used. In summary the resolution of a DFT depends on the length and shape of the window used, and is independent of the data being processed.

In addition to limiting spectral resolution an N-point window has another effect. Note from Fig. 5.1a the decaying sidelobes of the window squared magnitude spectrum. These sidelobes will bias the amplitude of adjacent frequency bins when $W(\mathtt{e}^{j\omega})$ is convolved with the DTFT of the sampled signal. Moreover, since the DTFT of the sampled signal is periodic, the sidelobes of the window cause spectral energy from the spectral replicas of the signal to mix into adjacent replica bins. This effect is referred to as leakage.

Problem 1

In this discussion the effect of window type on resolution and leakage is examined briefly. Consider the two windows

- Rectangular Window

$$\alpha[n] = \begin{cases} 1 & n = 0, 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

- Hamming Window (N even)

$$\beta[n] = \begin{cases} .54 + .46\mathtt{cos}(\frac{2\pi n}{N}) & -\frac{N}{2} \leq n \leq \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$$

1. Let $N = 10$, theoretically evaluate and plot the squared magnitude in dBs of the DTFT associated with each of the windows. Indicate on the plots where the mainlobe and sidelobes are.

2. Compare each window. Which window would yield better resolution? Which window exhibits lower sidelobes? Justify your answers.

3. How will increasing the window length N affect resolution and leakage?

FIR Filter Design - Windowing

In the course of this laboratory, you will be required to design numerous filters.To properly present your design overlay the frequency response of the designed filter over that of an ideal filter given the specifications. An example of this is shown in Fig. 5.2.

In the process of filter design a desired frequency response is specified and the problem is to determine the time-domain impulse function that achieves this desired response. Unfortunately in most cases the ideal frequency response is unobtainable in practice. To illustrate this limitation consider the ideal lowpass filter specified by

$$H(\mathtt{e}^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

where ω_c denotes the cutoff frequency of the filter that delineates the passband and stopband regions of the ideal filter. Since the impulse response of the ideal filter does not have a uniformly convergent DTFT, the actual filtering operation cannot be performed without windowing the sequence $h[n]$. Recall from our discussion concerning the effect of windowing on the resolution of the DFT sequence. In the same context the windowing of the ideal impulse response results in the smearing of the filter response, with the following effects

- **A broadening of the transition region between the passband and stopband regions.**
- **The appearance of ripples in the passband due to Gibbs oscillations.**
- **Non-ideal attenuation in the stopband.**

The design of a *finite impulsive response* (FIR) filter by windowing can be viewed as a tradeoff between the width of the transition region and stopband attenuation. The approach consists of selecting a N point window sequence $\{w[n]\}$ which will yield an acceptable approximation to the ideal filter response. To quantify what we mean by an acceptable approximation, we consider some type of optimality criteria. For instance consider the least-square or mean-square error criteria. Based on these criteria we seek a sequence window sequence $\{w[n]\}$ so that

$$H_w(\mathrm{e}^{j\omega}) = T_s \sum_{n=-\infty}^{\infty} h[n]w[n]\mathrm{e}^{-j\omega nT_s} = T_s \sum_{n=0}^{N-1} h_w[n]\mathrm{e}^{-j\omega nT_s}$$

is the best mean square error fit to the ideal frequency response

$$H(\mathrm{e}^{j\omega}) = T_s \sum_{n=-\infty}^{\infty} h[n]\mathrm{e}^{-j\omega nT_s}$$

where

$$\epsilon^2 = \int_{\omega} |H(\mathrm{e}^{j\omega}) - H_w(\mathrm{e}^{j\omega})|^2 d\omega$$

denotes mean-square error. Based on properties associated with the least-square convergence of the Fourier series it can be shown that the rectangular window

$$w[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

satisfies the mean-square error optimality criteria. We note that although the rectangular window provides the sharpest possible transition region, attenuation in the stopband is generally unsatisfactory for most applications. In this regard by trading off transition region width with a window of a more gradual tapering one can increase stopband rejection.

In the course of designing a FIR filter by the window approach exact passband and stopband frequencies cannot be specified because of the smearing associated with windowing the ideal response. In order to complete the design, the determination of the final filter impulse response must be derived by a trial and error procedure.

Problem 2

One possible application of designing `FIR` filters by windowing is the approximation of an *infinite impulse response* (`IIR`) filter by a `FIR` structure. In this problem we consider deriving such an approximation based on the following windows

- **Rectangular Window**

$$\alpha_n = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

- **Hanning Window**

$$\beta_n = \begin{cases} 0.5 + 0.5\texttt{cos}(\frac{2\pi n}{N-1}) & |n| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

- **Bartlett (Triangular) Window**

$$\gamma_n = \begin{cases} 1 - \frac{2|n|}{N-1} & |n| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

These windows will be used to synthesize an N-point `FIR` approximation for the following two systems. The first system has the impulse response

$$h_1[n] = \begin{cases} 4\frac{\texttt{sin}^2(\pi 0.05(n-L))}{\pi 0.05(n-L)^2} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with the second system having the form

$$h_2[n] = \begin{cases} 5.0\texttt{e}^{-0.4(n-L)} & n \geq L \\ 0 & \text{otherwise} \end{cases}$$

Note that the sampling period is taken to be $T_s = 1$ second.

1. Plot the window sequences α_n, β_n and γ_n, overlaying each sequence onto the same plot. Consider a window length of 27 samples. Label each window, and use a different line style to distinguish each sequence. Plot the squared magnitude in `dB` of the `DTFT` corresponding to each window over the normalized frequency interval $0 \leq \omega < \pi$. Which window would result in a filter design with the sharpest transition region? Likewise, which window would result in a filter design with the lowest stopband attenuation?

2. Plot the impulse response associated with $h_1[n]$ and $h_2[n]$ for $L = 64$, over the time interval $-64, -63, \ldots, 63, \ldots, 127, 128$, using the **comb** routine. Approximate by a FIR filter of length 27 samples, the impulse responses $h_1[n]$ and $h_2[n]$ using the three window sequences

α_n, β_n and γ_n. Explain how you intend to window the impulse responses. Would you expect that the resulting FIR filters have linear phase? Plot the frequency magnitude response in dB for each of the FIR filter designs. Discuss the relative stopband attenuation and transition region width of each design.

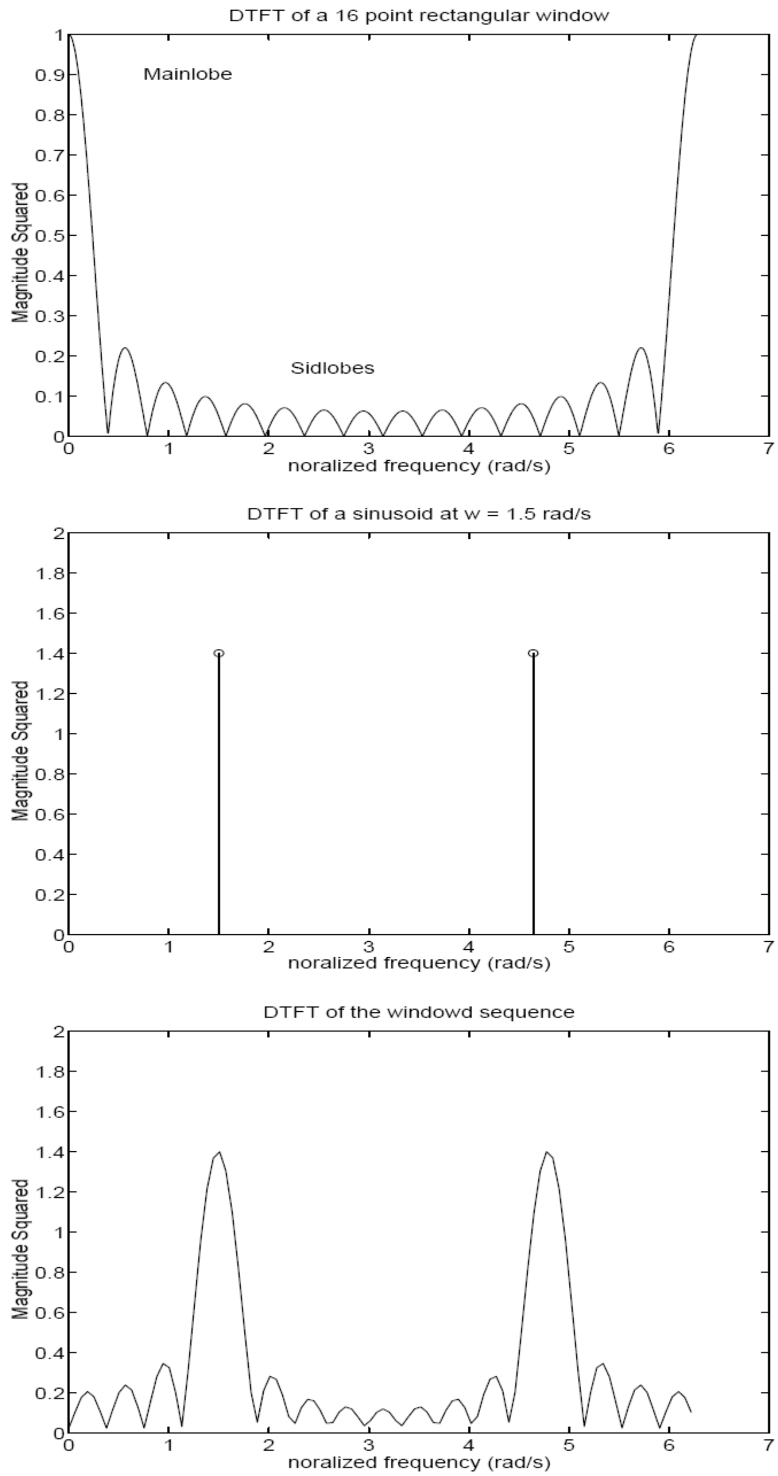

Figure 5.1: Effects of rectangular windowing in the frequency domain

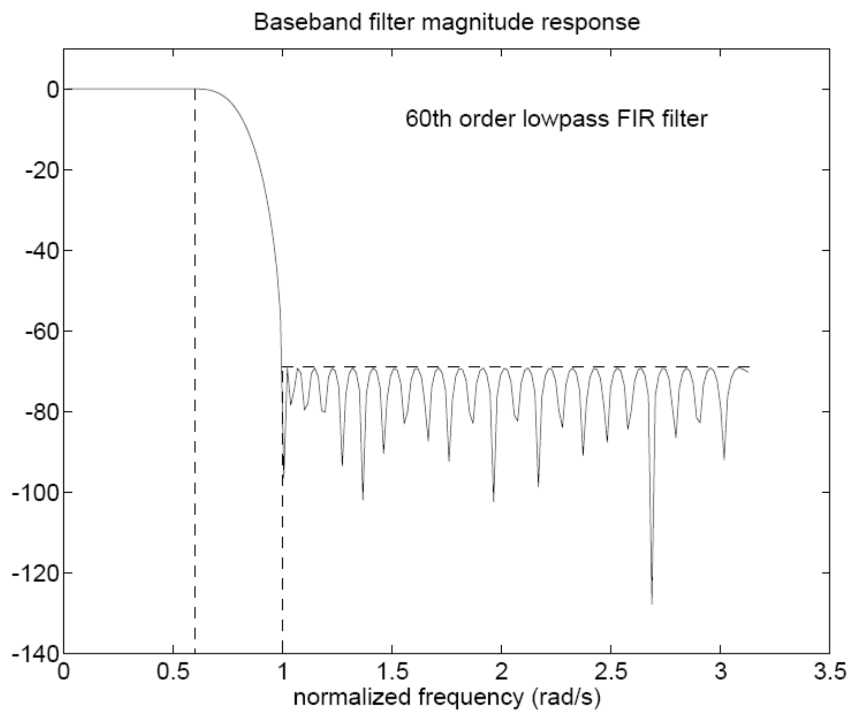

Figure 5.2: Desired and Designed filter response