

# COMP 371 -- Winter 2012

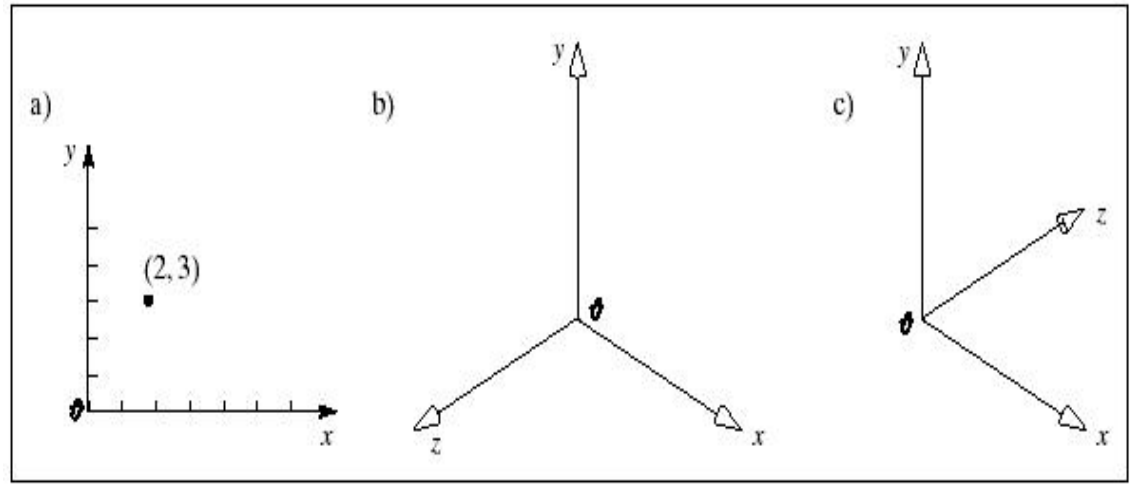
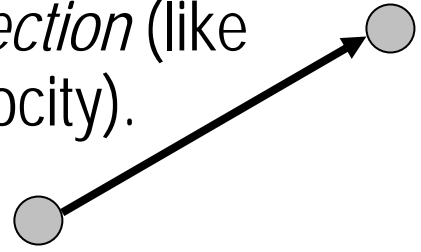
## Computer Graphics

- Matrix and Vector Operations
- 2D/3D Graphics Concepts



# Basic definitions

- Points specify *location* in space (or in the plane).
- Vectors have *magnitude* and *direction* (like velocity).

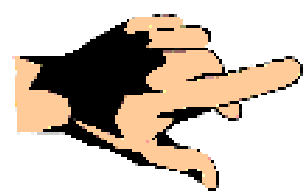


a) Two-dimensional

b) *Right-handed*  
 Math, engineering, physics, ...  
 Set up models for objects (local and world spaces)

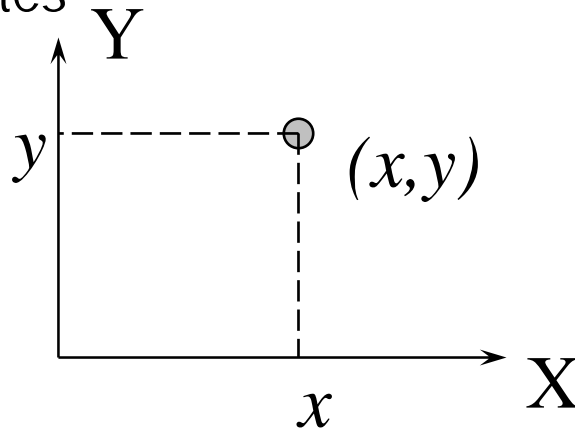


c) *Left-handed*  
 Natural in computer graphics  
 Viewing systems and "cameras"  
 (view space)

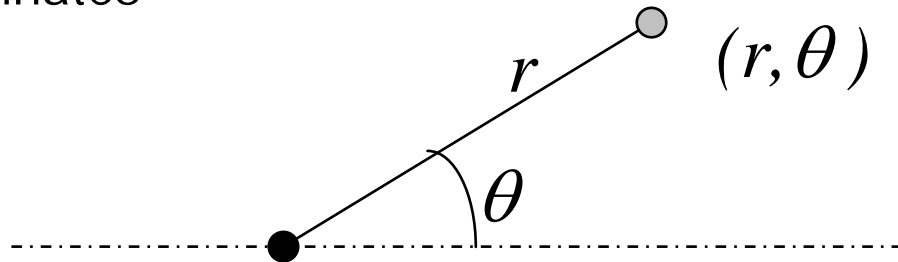


# 2D Co-ordinate Systems

- Concept of Dimension
  - Dimension of Space v/s Dimension of Object
- Cartesian Co-ordinates



- Polar Co-ordinates



# Basic Geometry: Objectives

- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
  - Line segments
  - Polygons



# Basic Elements

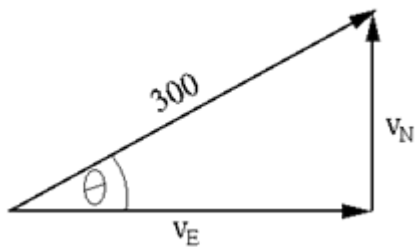
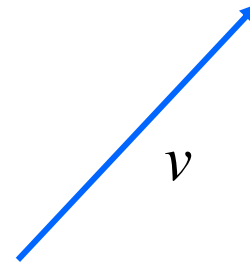
- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

# Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- **Scalars** can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- **Scalars** alone have no geometric properties
- **Scalar**: variable described by a single number (magnitude)
  - Temperature = 20 °C
  - Density = 1 g/cm<sup>3</sup>
  - Image intensity (pixel value)

# Vectors

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



$$\mathbf{v} = \begin{bmatrix} v_n \\ v_e \end{bmatrix}$$

Row vector

$$\mathbf{d} = [3 \quad 4 \quad 9]$$

Column vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

# What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

$m \times n$  matrix

$n$  columns

$$m \text{ rows} \left\{ \begin{array}{cc} a_{00} & a_{01} \\ a_{10} & a_{11} \end{array} \right.$$

# Basic Operations

- Transpose: Swap rows with columns

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad M^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad V^T = [x \quad y \quad z]$$

# Basic Operations

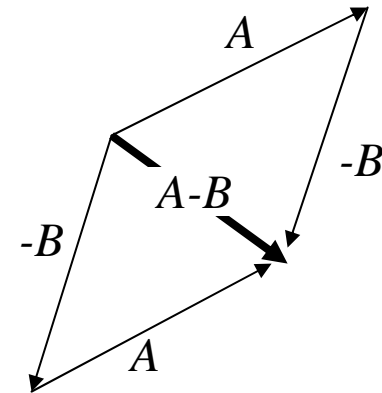
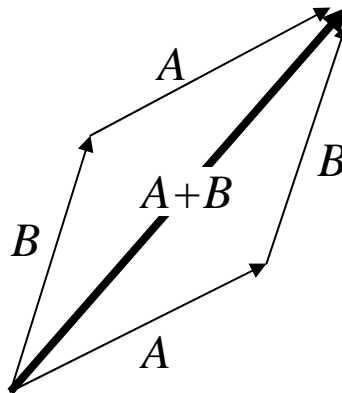
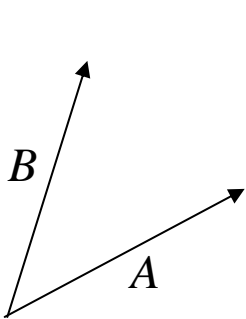
- Addition and Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

**Just add elements**

**Just subtract elements**



# Basic Operations

- Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \quad \text{Multiply each row} \\ \text{by each column}$$

An  $m \times n$  can be multiplied by an  $n \times p$  matrix to yield an  $m \times p$  result

- Is  $AB = BA$ ? Maybe, but maybe not!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \dots \\ \dots & \dots \end{bmatrix} \quad \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & \dots \\ \dots & \dots \end{bmatrix}$$

# Determinant of a Matrix

- Used for inversion
- If  $\det(A) = 0$ , then  $A$  has no inverse
- Can be found using factorials, pivots, and cofactors!
- Lots of interpretations

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

# Matrix Inverse

- Inverse of Matrix: A
  - This is usually used while solving a set of linear equations:
  - $Ax = B$
  - $x = A^{-1} B$

- Inverse formula:

- 2D matrix:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- 3D matrix:

$A_{ik}$  is the cofactor of element  $a_{ik}$ , i. e.,

$A_{ij} = (-1)^{i+k} B_{ik}$  and  $B_{ik}$  is the minor of  $a_{ik}$  i.e., the determinant of partial matrix obtained by removing  $i$ th row and  $k$ th column from A.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

# Determinant of a Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

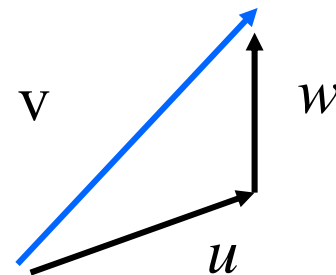
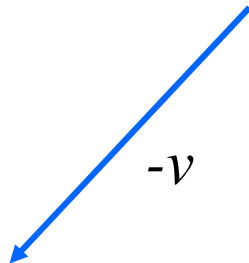
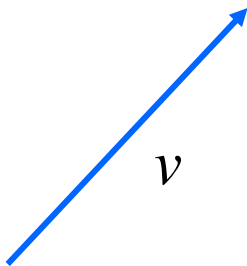
$$\begin{vmatrix} a & b & c & a & b & c \\ d & e & f & d & e & f \\ g & h & i & g & h & i \end{vmatrix}$$

For a 3×3 matrix:  
Sum from left to right  
Subtract from right to left

**Note:** In the general case, the determinant has  $n!$  terms

# Vector Operations

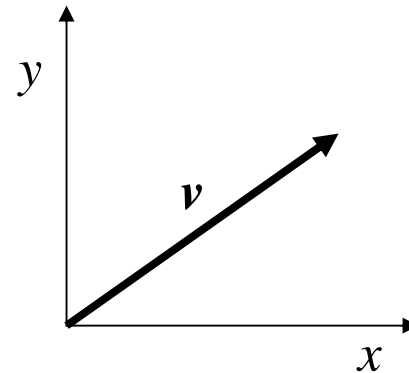
- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



# Vector Operations

- Vector:  $n \times 1$  matrix
- Interpretation: a point or line in  $n$ -dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

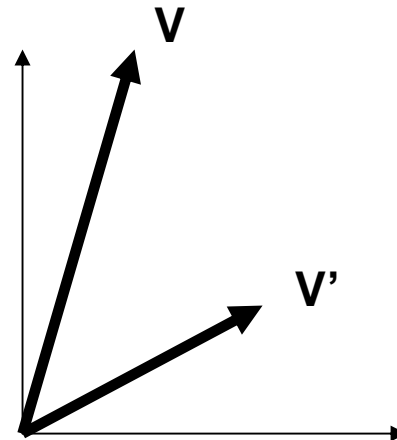
$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



# Vector Interpretation

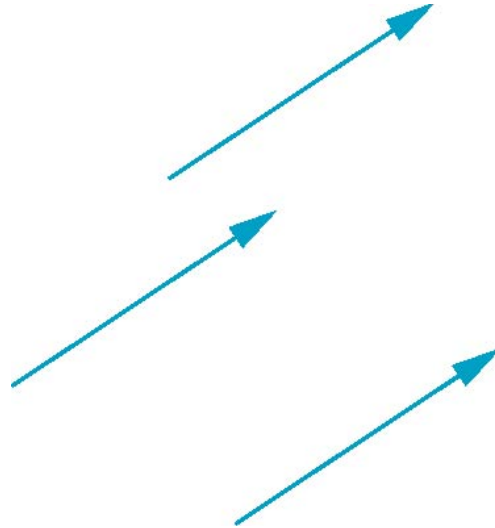
- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# Vectors Lack Position

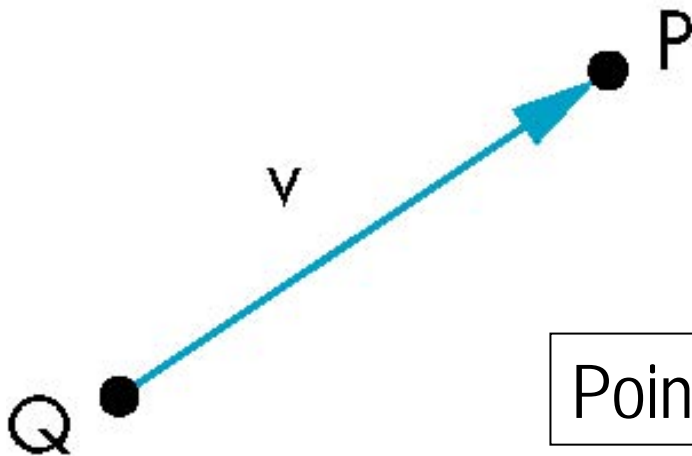
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

# Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition
- Points specify *location* in space (or in the plane).
- Vectors have *magnitude* and *direction* (like velocity).



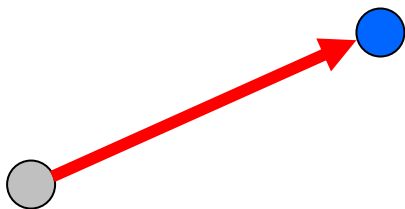
$$v = P - Q$$

$$P = v + Q$$

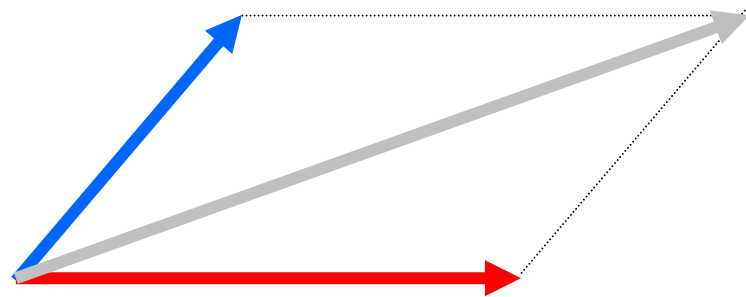
Points  $\neq$  Vectors

# More on points and vectors

- Point + vector = point
- vector + vector = vector



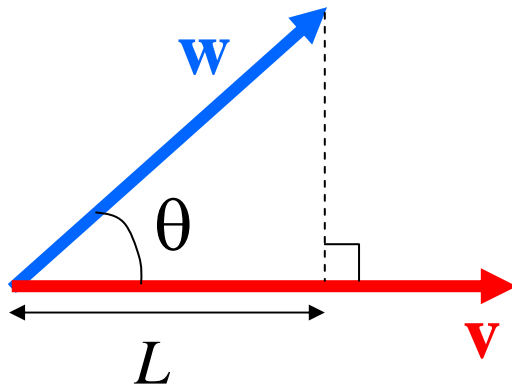
Parallelogram rule



# Inner (dot) product

- Defined for vectors:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \theta$$

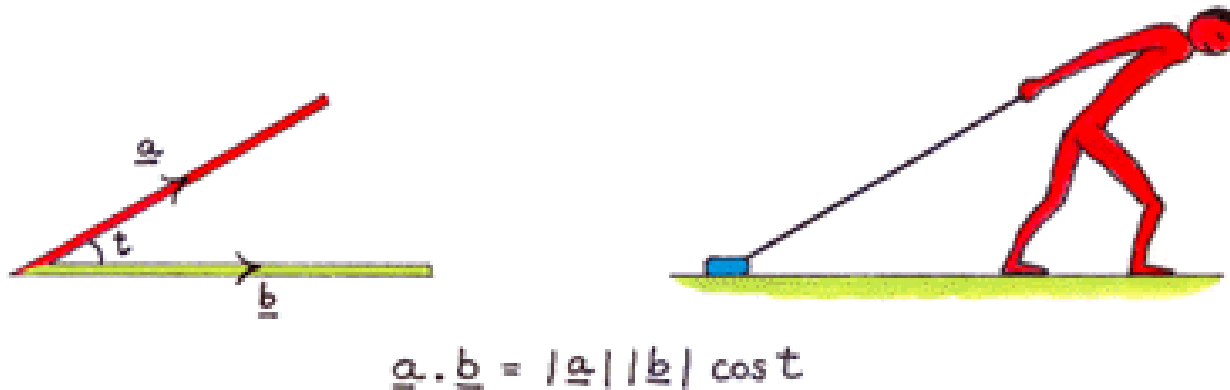


$$\cos \theta = \frac{L}{\|\mathbf{w}\|}$$

$$L = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|}$$

Projection of  $\mathbf{W}$  onto  $\mathbf{V}$

# Dot product: interpretation

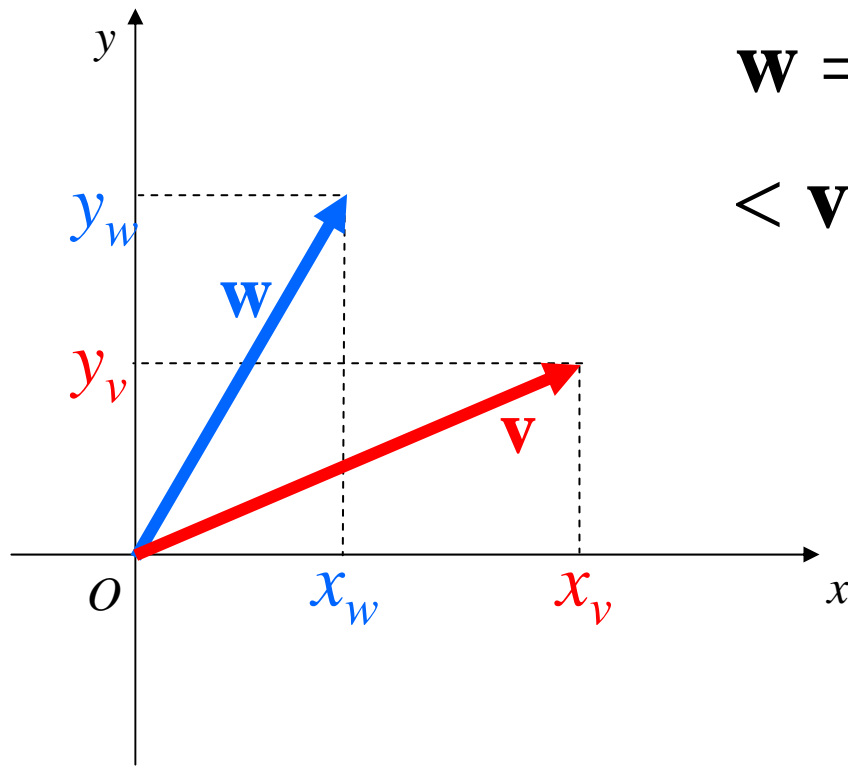


The man is pulling the block with a constant force  $\mathbf{a}$  so that it moves along the horizontal ground. The **work done** in moving the block through a distance  $\mathbf{b}$  is then given by the distance moved through multiplied by the magnitude of the component of the force in the direction of motion. This is  $|\mathbf{a}| |\mathbf{b}| \cos t$  so we define the dot product as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos t$$

where  $t$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  when they are placed tail to tail.

# Dot product in coordinates (2D)



$$\mathbf{v} = (x_v, y_v)$$

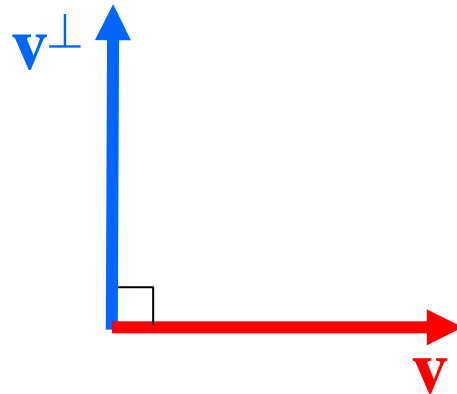
$$\mathbf{w} = (x_w, y_w)$$

$$\langle \mathbf{v}, \mathbf{w} \rangle = x_v x_w + y_v y_w$$

# Perpendicular vectors (2D)

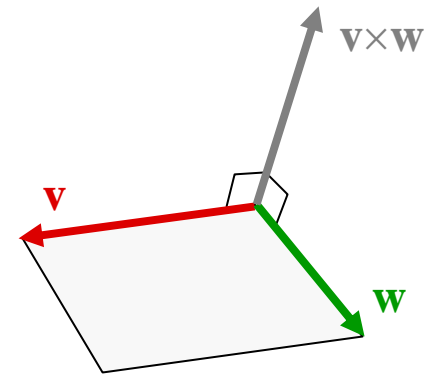
$$\langle \mathbf{v}, \mathbf{w} \rangle = 0$$

$$\mathbf{v} = (x_v, y_v) \Rightarrow \mathbf{v}^\perp = \pm(-y_v, x_v)$$



# Cross product

- $\mathbf{v} \times \mathbf{w}$  is a new vector:
  - $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| |\sin \angle(\mathbf{v}, \mathbf{w})|$
  - $(\mathbf{v} \times \mathbf{w}) \perp \mathbf{v}$ ,  $(\mathbf{v} \times \mathbf{w}) \perp \mathbf{w}$
  - $\mathbf{v}$ ,  $\mathbf{w}$ ,  $\mathbf{v} \times \mathbf{w}$  is a right-hand system



- In matrix form in 3D:

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^T, \quad \mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)^T$$

$$\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{pmatrix} = (\mathbf{v}_2 \mathbf{w}_3 - \mathbf{v}_3 \mathbf{w}_2, \mathbf{w}_1 \mathbf{v}_3 - \mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_1 \mathbf{w}_2 - \mathbf{v}_2 \mathbf{w}_1)$$

# Cross Product (cont.)

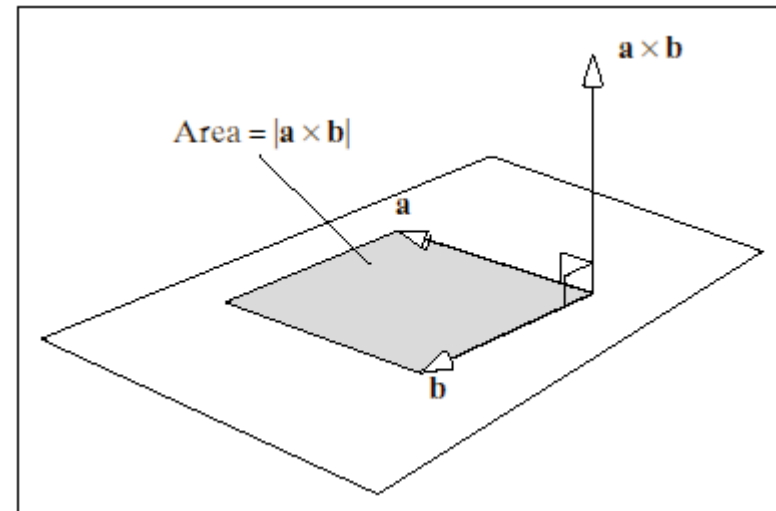
- Definition: If  $\mathbf{a} = (x_a, y_a, z_a)^T$  and  $\mathbf{b} = (x_b, y_b, z_b)^T$ , then:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)^T$$

- $\mathbf{c}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$  (direction given by right-hand rule), with magnitude given by

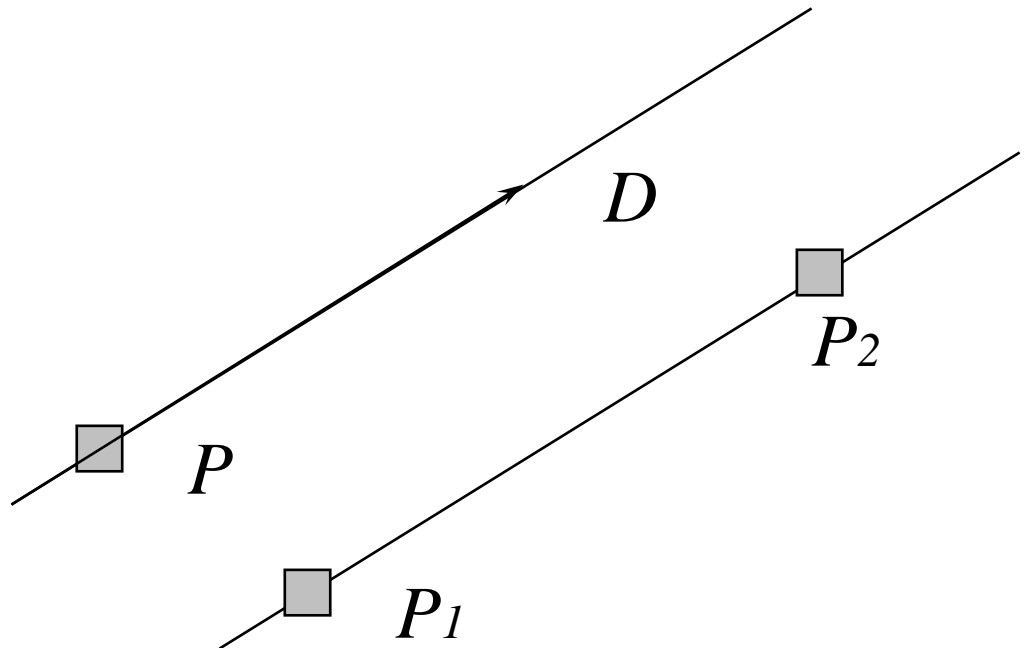
$$|\mathbf{c}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

$\theta$  — angle between  $\mathbf{a}$  and  $\mathbf{b}$



# Line

- Representation  
(Start Point, End Point)  
or (Point, Direction)
- Equation of Line  
 $L = P + t * D$   
or  
 $L = P_1 + t * (P_2 - P_1)$



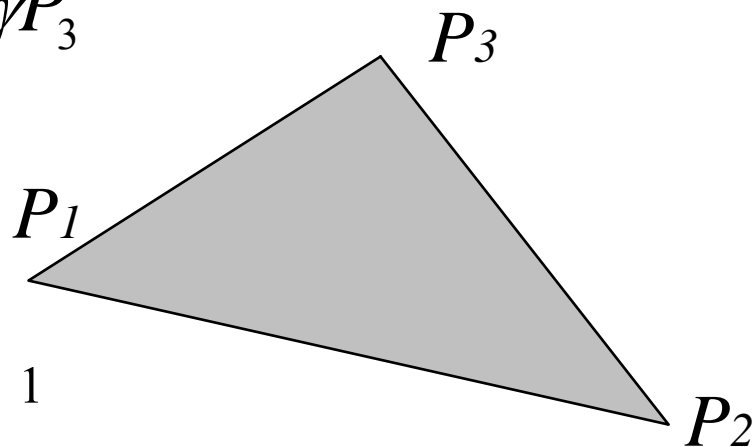
# Barycentric Co-ordinates

- Any point on a plane can be written as affine combination of three distinct points

$$P = \alpha P_1 + \beta P_2 + \gamma P_3$$

where,

$$\alpha + \beta + \gamma = 1$$



Note that, if  $0 < (\alpha, \beta, \gamma) < 1$   
point P lies inside the triangle

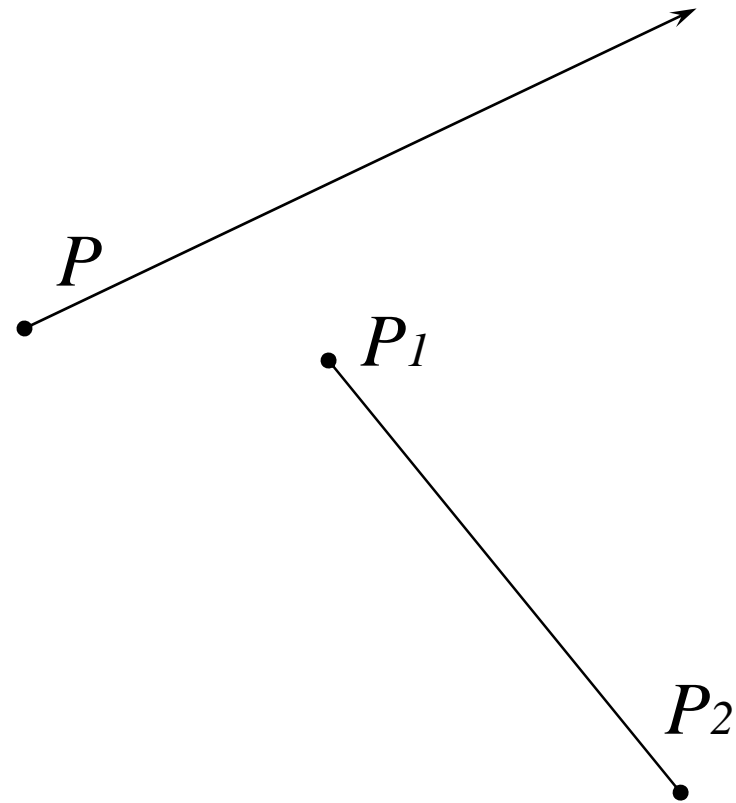
# Ray and Segments

- A Ray is a line with

$$0 \leq t \leq \infty$$

- A segment is a line with

$$0 \leq t \leq 1$$



# Other Simple Objects

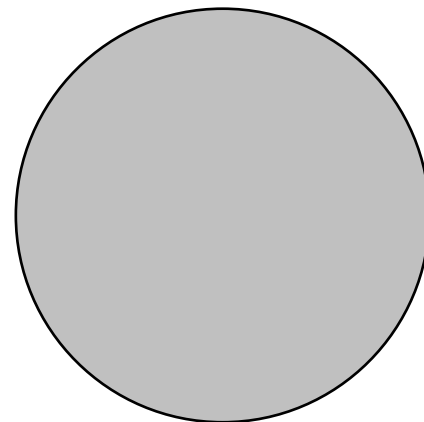
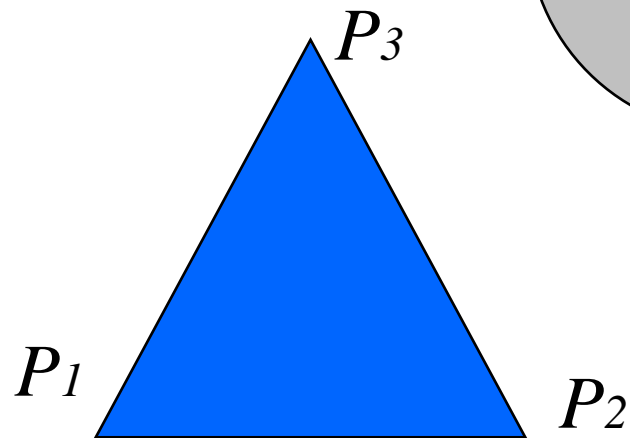
- Circle

$$x^2 + y^2 = R^2$$

- Triangle

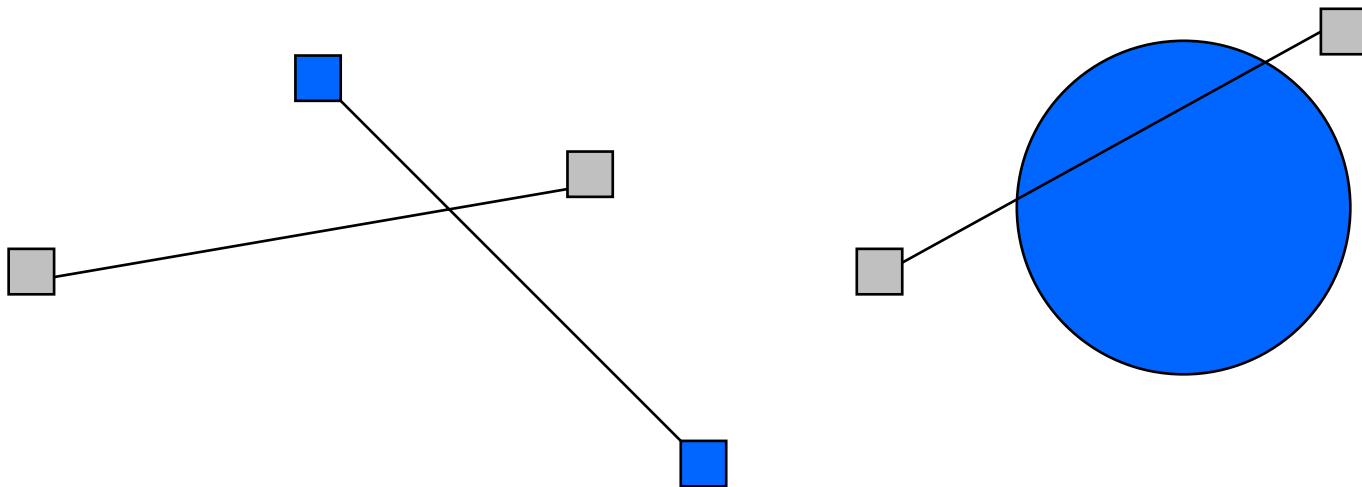
Inside Points

Area



# Intersections

- Finding intersections between various geometric objects is one of the important task in Computer Graphics algorithms



# Area Computations

- Finding lengths, areas, volumes, etc.. of various geometric objects is another important problem in Computer Graphics

How to find an area of a polygon in 2D?

Hint:

