

Concordia University  
COMP 371 --- Winter 2012  
**Computer Graphics**  
Quiz Review Exercises

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**PURPOSE:** This practice set will give you the opportunity to learn more about 2D and 3D transformations.

**Question #1:**

Consider the following frequently used 2-D transformations:

- 1) Translation:  $x' = x + s$   $y' = y + t$   
The point  $(x,y)$  is moved by vector  $(s,t)$
- 2) Scaling:  $x' = ax$   $y' = by$   
 $x$ -axis distances are magnified by  $a$ ;  $y$ -axis distances are magnified by  $b$
- 3) Rotation:  $x' = x \cos \theta - y \sin \theta$   $y' = x \sin \theta + y \cos \theta$   
The point  $(x,y)$  is rotated through an angle  $\theta$  about  $(0,0)$
- 4) Reflection:  $x' = x$   $y' = -y$   
The point  $(x,y)$  is reflected about the  $x$ -axis
- 5) Shear:  $x' = x + hy$   $y' = y$   
The point  $(x,y)$  is shifted in the  $x$ -direction by the amount  $hy$

- a) Express the above transformation in matrix form in homogenous coordinates.
- b) Determine the form of a transformation matrix for a rotation by an angle  $t$  about the point  $(e,f)$  followed by a translation by the vector  $(c,d)$  followed by a scaling with respect to  $(e,f)$ .
- c) Prove that a rotation by an angle  $\theta_1$  followed by a rotation by an angle  $\theta_2$  is equivalent to a single rotation by an angle  $\theta_1 + \theta_2$ .
- d) Two transformations that give the same final result regardless of the order in which they are applied are called commutative. Prove that rotation followed by a rotation are commutative operations. Prove that translation and scaling are not commutative. Likewise, prove that rotation and scaling are not commutative unless  $a=b$ .
- e) Determine the transformation matrix for a reflection about an arbitrary line  $y = mx + b$ .

**Question #2:**

- a) Derive a 4x4 matrix that will rotate by 60 degrees points about an axis in the direction (3,2,5) that goes through the point (1,2,3). You should be able to apply this matrix to points written in homogenous coordinates. Show your work.
- b) The projection matrix constructed by the call `glFrustum(l, r, b, t, n, f)` is

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

1. Find view coordinates X, Y, and Z of point  $p=(x,y,z)$ .
  2. Consider the special case in which  $l=b=-1$  and  $r=t=1$ . Find X, Y, and Z for  $p=(x,y,-n)$  and  $p=(x,y,-f)$ .
- c) Find the matrix computed by `gluPerspective(alpha, aspect_ratio, near, far)` using the matrix computed by `glFrustum()` in the preceding question.
- d) Given an eye position at  $e=(3, 3, 1)$ , a look-at point  $a=(3, 4, 1)$  and an up-vector  $\vec{v}_{up} = (1,0,0)$ , construct the view transformation matrix that maps the world space points to the view space points. Determine the view space coordinates of the following world space points.
1. (3,1,2)
  2. (4,1,-1)
  3. (0,0,0)

DEPARTMENT OF COMPUTER SCIENCE  
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COMP 371/4: Computer Graphics

**Theory Questions Solutions:**

**Question 1**

$$x' = x + s \qquad y' = y + t \qquad (1)$$

$$x' = ax \qquad y' = by \qquad (2)$$

$$x' = x \cos \theta - y \sin \theta \qquad y' = x \sin \theta + y \cos \theta \qquad (3)$$

$$x' = x \qquad y' = -y \qquad (4)$$

$$x' = x + hy \qquad y' = y \qquad (5)$$

a) In matrix form in homogeneous coordinates:

(1):

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(2):

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(3):

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(4):

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(5):

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

b) Composite Transformation matrix:

$$\begin{aligned}
& T(e, f) \cdot S(a, b) \cdot T(-e, -f) \cdot T(c, d) \cdot T(e, f) \cdot R(t) \cdot T(-e, -f) = \\
& \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \\
& \bullet \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix} = \\
& \begin{bmatrix} a & 0 & a(c+e) \\ 0 & b & b(d+f) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & -e \cos t + f \sin t \\ \sin t & \cos t & -e \sin t - f \cos t \\ 0 & 0 & 1 \end{bmatrix} = \\
& \begin{bmatrix} a \cos t & -a \sin t & -ae \cos t + af \sin t + ac + e \\ b \sin t & b \cos t & -be \sin t - bf \cos t + bd + f \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

c) Composite Rotation Transformation matrices:

$$\begin{aligned}
R(\theta_1) \cdot R(\theta_2) &= \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
& \begin{bmatrix} \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 & -(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) & 0 \\ \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 & -\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
& \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R(\theta_1 + \theta_2)
\end{aligned}$$

d) Mixed Composite Transformation matrices:

- Rotation and Rotation (see part(c)).
- Scale and Translation.

$$\begin{aligned}
 & T(e, f) \cdot S(a, b) && S(a, b) \cdot T(e, f) \\
 & \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} && \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \\
 & \begin{bmatrix} a & 0 & e \\ 0 & b & f \\ 0 & 0 & 1 \end{bmatrix} && \begin{bmatrix} a & 0 & ae \\ 0 & b & bf \\ 0 & 0 & 1 \end{bmatrix} \\
 & && \neq
 \end{aligned}$$

Not commutative (even if  $a = b$ , unless  $a = 1$  which would just be the identity matrix).

- Scale and Rotation.

$$\begin{aligned}
 & S(a, b) \cdot R(\theta) && R(\theta) \cdot S(a, b) \\
 & \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} && \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 & \begin{bmatrix} a \cos \theta & -a \sin \theta & 0 \\ b \sin \theta & b \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} && \begin{bmatrix} a \cos \theta & -b \sin \theta & 0 \\ a \sin \theta & b \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 & && \neq
 \end{aligned}$$

Not commutative unless  $a = b$ .

e) We accomplish the transformation by translating the line  $y = mx + b$  to the origin followed by rotation through angle  $-\theta$ , mirror reflection about x-axis and by "undoing" translation and rotation. The transformation matrix is

$$\begin{aligned}
 M_{(m,b)} &= T_{(0,b)} \cdot R_{(0,0)}(\theta) \cdot RE_{(1,0)} \cdot R_{(0,0)}(-\theta) \cdot T_{(0,-b)} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & 2 \sin \theta \cos \theta & -2b \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \sin^2 \theta - \cos^2 \theta & b - b(\sin^2 \theta - \cos^2 \theta) \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos 2\theta & \sin 2\theta & -b \sin 2\theta \\ \sin 2\theta & -\cos 2\theta & b(1 + \cos 2\theta) \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

$$= \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & \frac{-2mb}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} & \frac{2b}{1+m^2} \\ 0 & 0 & 1 \end{bmatrix}$$

where the last equality follows from  $\tan \theta = m$ ,  $\sin \theta = \frac{m}{\sqrt{1+m^2}}$ ,  $\cos \theta = \frac{1}{\sqrt{1+m^2}}$ ,  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{1-m^2}{1+m^2}$ ,  $b(1 + \cos 2\theta) = b(1 + \frac{1-m^2}{1+m^2}) = \frac{2b}{1+m^2}$ .

**Question 2**

(a)

We need to determine

$$M = T(1, 2, 3) \cdot M_R(\theta) \cdot T(-1, -2, -3)$$

where  $M_R(\theta)$  is the rotation matrix for rotation by angle  $\theta = 60^\circ$  about axis in the normalized direction  $v = (3, 2, 5)$ , where  $\|v\| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$ . So when  $v$  is normalized, we get

$$u = \frac{v}{\|v\|} = (u_x, u_y, u_z) = \frac{(3, 2, 5)}{\sqrt{38}}.$$

The rotation matrix

$$M_R(\theta) = \begin{pmatrix} u_x^2 + (1 - u_x^2) \cos \theta & u_x u_y (1 - \cos \theta) - u_z \sin \theta & u_x u_z (1 - \cos \theta) + u_y \sin \theta & 0 \\ u_x u_y (1 - \cos \theta) + u_z \sin \theta & u_y^2 + (1 - u_y^2) \cos \theta & u_y u_z (1 - \cos \theta) + u_x \sin \theta & 0 \\ u_x u_z (1 - \cos \theta) + u_y \sin \theta & u_y u_z (1 - \cos \theta) + u_x \sin \theta & u_z^2 + (1 - u_z^2) \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

can be obtained using quaternions. Substituting the values we obtain the following transformation matrix

$$M = \begin{bmatrix} 0.618 & -0.623 & 0.478 & 0.194 \\ 0.86 & 0.39 & -0.273 & 1.17 \\ -0.114 & 0.149 & 0.872 & -0.028 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) 1. Let  $p$  be a point that has been transformed by the model-view matrix and let  $M$  be the matrix produced by  $glFrustum(l, r, t, b, n, f)$  call. We compute  $X$ ,  $Y$ , and  $Z$  as follows:

$$\begin{aligned} q &= M \cdot p \\ X &= q_1/q_4 \\ Y &= q_2/q_4 \\ Z &= q_3/q_4 \end{aligned}$$

$X$  and  $Y$  are the view coordinates of the point  $p$  after projection. After plugging in  $M$  we obtain

$$\begin{aligned} X &= \frac{2nx}{z(l-r)} + \frac{l+r}{l-r} \\ Y &= \frac{2ny}{z(b-t)} + \frac{t+b}{t-b} \\ Z &= \frac{z(f+n) + 2fn}{z(f-n)} \end{aligned}$$

2.

With the values given, the matrix  $M$  becomes

$$M = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Plugging in  $p$  we obtain

$$(i) \quad M \cdot \begin{bmatrix} x \\ y \\ -n \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -1 \\ 1 \end{bmatrix}.$$

$$(ii) \quad M \cdot \begin{bmatrix} x \\ y \\ -f \\ 1 \end{bmatrix} = \begin{bmatrix} nx/f \\ ny/f \\ 1 \\ 1 \end{bmatrix}.$$

- Result (i) shows that when  $z = -n$ , the  $x$  and  $y$  values are unchanged. We can deduce that the “near” plane is actually the plane of the screen.
- From (i) and (ii), we can deduce that the frustum transformation transforms the viewing volume into a cube bounded by  $|x| \leq 1$ ,  $|y| \leq 1$ , and  $|z| \leq 1$ . This simplifies clipping, because any object with a coordinate greater than 1 need not be rendered.

(c)

Figure 1 shows the effect of the `gluPerspective()` matrix. From this diagram and the previous question, we know that the screen is the near plane. The perspective transformation is not “skewed” — the viewer is looking directly at the centre of the screen. Thus we have

$$\begin{aligned} h &= 2 \times near \times \tan \frac{1}{2}\alpha \\ w &= \rho h \end{aligned}$$

where  $\rho$  is the aspect ratio. The arguments of an equivalent invocation of `glFrustum()` would be

$$\begin{aligned} t &= h/2 \\ b &= -h/2 \\ r &= \rho h/2 \\ l &= -\rho h/2 \end{aligned}$$

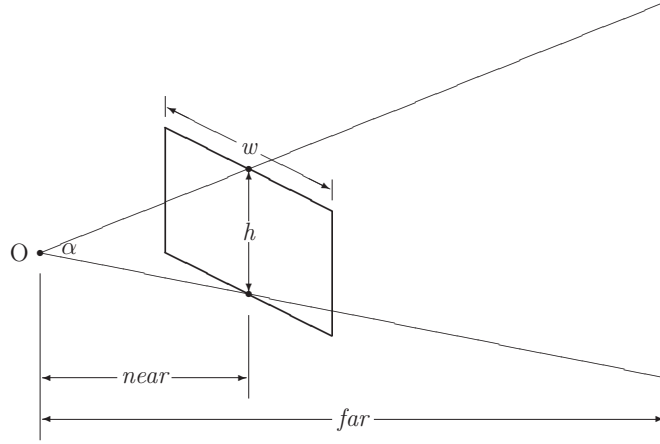


Figure 1: `gluPerspective(alpha, ar, near, far)`

Consequently, the transformation matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho \tan \frac{1}{2}\alpha & 0 & 0 & 0 \\ 0 & \frac{1}{\tan \frac{1}{2}\alpha} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(d)

We need to determine  $M_{WC \rightarrow VC} = R \cdot T$ . First,

$$T = \begin{bmatrix} 1 & 0 & 0 & -\text{eye}_x \\ 0 & 1 & 0 & -\text{eye}_y \\ 0 & 0 & 1 & -\text{eye}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also,  $N = \text{eye} - \text{look-at} = e - a = (3 - 3, 3 - 4, 1 - 1) = (0, -1, 0)$ . Thus  $\hat{n} = N/\|N\| = (0, -1, 0)$ . And  $V = v_{\text{up}} = (1, 0, 0)$ , so

$$V \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ N_x & N_y & N_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (-1)\hat{k} = (0, 0, -1).$$

So,  $\hat{u} = V \times N / \|V \times N\| = (0, 0, -1)$ . Finally,

$$\hat{v} = \hat{n} \times \hat{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1\hat{i} + 0\hat{j} + 0\hat{k} = (1, 0, 0).$$

Therefore,

$$R = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This gives a final view transformation of

$$M_{WC \rightarrow VC} = R \cdot T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.

$$R \cdot T \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

2.

$$R \cdot T \begin{bmatrix} 4 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

3.

$$R \cdot T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \\ 1 \end{bmatrix}.$$