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Sylvester–Gallai Theorem and Metric Betweenness

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Abstract. Sylvester conjectured in 1893 and Gallai proved some 40 years later that every finite set S of points in the plane includes two points such that the line passing through them includes either no other point of S or all other points of S. There are several ways of extending the notion of lines from Euclidean spaces to arbitrary metric spaces. We present one of them and conjecture that, with lines in metric spaces defined in this way, the Sylvester–Gallai theorem generalizes as follows: in every finite metric space there is a line consisting of either two points or all the points of the space. Then we present meagre evidence in support of this rash conjecture and finally we discuss the underlying ternary relation of metric betweenness.

1. The Sylvester–Gallai Theorem

In March 1893 Sylvester [33] proposed the following problem:

> *Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third*, *unless they all lie in the same right line*.

The May 1893 issue of the same journal reports a four-line "solution" proposed by H.J. Woodall, A.R.C.S., followed by a comment pointing out two flaws in the argument and sketching another line of enquiry, which "... is equally incomplete, but may be worth notice." Some 40 years later, Erdős revived the problem and the first proof was found shortly afterwards by T. Gallai (named Grünwald at that time): see [15]. Additional proofs were given by R.C. Buck, N.E. Steenrod, and R. Steinberg; in particular, Steinberg's proof may be seen as a projective variation on Gallai's affine theme; Coxeter [10], [11, Section 12.3] transformed it into an even more elementary form. The theorem also follows, through projective duality, from a result of Melchior [24], proved by a simple application of Euler's polyhedral formula. A very short proof was given by L.M. Kelly; this proof uses the notion of Euclidean distance and can be found in [10], [11, Section 4.7] and [18, Chapter 8]. Further information on the Sylvester–Gallai theorem is provided in [6], [16], and [26].

2. Ordered Geometry

We follow the tradition of writing $[abc]$ to mean that point b lies between points a and c in the sense of being an interior point of the line segment with endpoints a and c. In his development of geometry, Euclid used the notion of betweenness only implicitly; its explicit axiomatization was first carried out by Pasch [28] and then gradually refined by Peano [29], Hilbert [19], and Veblen [34]. In the resulting system, as presented by Coxeter [11], two primitive notions of *points* and *betweenness* are linked by ten axioms, of which the first seven address planar geometry and the remaining three address spatial geometry. The first four of the seven planar axioms are:

(A1) There are at least two points.
(A2) If a and b are distinct points, then there is at least one point c for which $[abc]$.
(A3) If $[abc]$, then $a \neq c$.
(A4) If $[abc]$, then $[cba]$ but not $[bca]$.

The next two of the seven planar axioms involve the notion of a *line ab*, defined—for any two distinct points a and b—as

$$\{x : [xab]\} \cup \{a\} \cup \{x : [axb]\} \cup \{b\} \cup \{x : [abx]\}. \tag{1}$$

(A5) If c and d are distinct points on the line ab, then a is on the line cd.
(A6) If ab is a line, then there is a point c not on this line.

The last of the seven planar axioms uses the notion of a *triangle*, defined as three points not lying on the same line:

(A7) If abc is a triangle and if $[bcd]$ and $[cea]$, then there is, on the line de, a point f for which $[afb]$.

The Sylvester–Gallai theorem states that

> *every set V of at least two but finitely many points* (*in a Euclidean space*) *includes distinct points a, b such that the intersection of the line ab and V is $\{a, b\}$ or V*;

its statement involves only the notions of points and lines; lines can be defined in terms of points and their betweenness by (1); Coxeter's proof of the theorem relies only on the seven axioms (A1)—(A7) of *ordered geometry*. (In the same spirit of methodological purity, but in a different minimalist setting, Pambuccian [27] conjectures that the Sylvester–Gallai theorem needs no order at all, in the sense that it should remain true in affine planes over fields—even skew fields—of characteristic zero that are not quadratically closed.)

3. Sylvester–Gallai Theorem in Metric Spaces?

For every real number p such that $p \geq 1$, the ℓ_p*-norm* on $\mathbf{R}^m$ is defined by

$$||(z_1, z_2, \ldots, z_m)||_p = \left(\sum_{i=1}^{m} |z_i|^p \right)^{1/p};$$

if $\varphi\colon\ V \to \mathbf{R}^m$ is a one-to-one mapping, then the mapping $\rho\colon\ (V \times V) \to \mathbf{R}$ defined by

$$\rho(x, y) = ||\varphi(x) - \varphi(y)||_p$$

is a metric and is called an ℓ_p*-metric*. In these terms, the Sylvester–Gallai theorem can be stated as follows.

Theorem 3.1. *Let (V, ρ) be a metric space such that $1 < |V| < \infty$; we write $[abc]$ to mean that a, b, c are three distinct points and*

$$\rho(a, b) + \rho(b, c) = \rho(a, c).$$

If ρ is an ℓ_2-metric, then V includes distinct points a, b such that the set

$$\{x : [xab]\} \cup \{a\} \cup \{x : [axb]\} \cup \{b\} \cup \{x : [abx]\}$$

is $\{a, b\}$ or V.

The assumption that ρ is an ℓ_2-metric—or at least some restrictive assumption about ρ—is crucial here: for example, if V consists of the five points

$$\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}$$

and ρ is defined by $\rho(x, y) = ||x - y||_1$, then each of the ten sets (1) consists of three or four points. Nevertheless, if no assumptions are placed on ρ in Theorem 3.1, but (1) is replaced by another set, identical with (1) when ρ is an ℓ_2-metric, then the resulting statement might remain valid. We define the substitute for (1) in four stages, gradually discarding information about ρ in the process.

First, given an arbitrary metric space (V, ρ), we define a ternary relation $\mathcal{B}(\rho)$ on V by

$$(a, b, c) \in \mathcal{B}(\rho) \quad \Leftrightarrow \quad a, b, c \text{ are all distinct and } \rho(a, b) + \rho(b, c) = \rho(a, c).$$

Next, discarding the order on each triple in $\mathcal{B}(\rho)$, we transform the ternary relation into a set $\mathcal{H}(\mathcal{B}(\rho))$ of three-point subsets of V:

$$\mathcal{H}(\mathcal{B}(\rho)) = \{\{a, b, c\} : (a, b, c) \in \mathcal{B}(\rho)\}.$$

Then we define a family $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ of subsets of V by

$$\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho))) = \{S \subseteq V : \text{no } T \text{ in } \mathcal{H}(\mathcal{B}(\rho)) \text{ has } |T \cap S| = 2\}.$$

If (V, ρ) is a Euclidean space, then $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ consists of all its affine subspaces; in general, $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ is closed under arbitrary intersection. (For certain other sets $\mathcal{H}$ of three-point subsets of a set V, unrelated to our $\mathcal{H}(\mathcal{B}(\rho))$, the same notion of $\mathcal{A}(\mathcal{H})$ has been investigated by Boros et al. [5]; they refer to sets in $\mathcal{A}(\mathcal{H})$ as *closed*.)

Finally, we define the *line* determined by distinct points a and b in V as the intersection of all members of $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ that contain $\{a, b\}$. This definition reflects the notion that the Euclidean line determined by two points is the affine hull of the two-point set.

Conjecture 3.2. *If* (V, ρ) *is a metric space such that* $1 < |V| < \infty$, *then* V *includes distinct points* a, b *such that the line* ab *is* $\{a, b\}$ *or* V.

4. Small Metric Spaces

Theorem 4.1. *If* (V, ρ) *is a metric space such that* $1 < |V| < 10$, *then* V *includes distinct points* a, b *such that the line* ab *is* $\{a, b\}$ *or* V.

Our proof of this result has two parts. The first part shows that every counterexample (V, ρ) to Theorem 4.1 must make $\mathcal{H}(\mathcal{B}(\rho))$ isomorphic to one of three particular families of three-point sets; the second part relies on a computer search to verify that none of these three families arises from any metric space (V, ρ) as its $\mathcal{H}(\mathcal{B}(\rho))$. We describe the details of both parts.

A 3-*uniform hypergraph* is an ordered pair $(V, \mathcal{H})$ such that V is a set and $\mathcal{H}$ is a family of distinct three-point subsets of V; elements of V are the *vertices* of the hypergraph and members of $\mathcal{H}$ are its *edges*. In each of our 3-uniform hypergraphs, all edges will be distinct.

Our definition of lines in a metric space (V, ρ) depends only on the 3-uniform hypergraph $(V, \mathcal{H}(\mathcal{B}(\rho)))$. We extend it to arbitrary finite 3-uniform hypergraphs $(V, \mathcal{H})$: for any two distinct points a, b of V, the *line ab* in the hypergraph is the return value of the program

$S = \{a, b\}$;
while some T in $\mathcal{H}$ has $|T \cap S| = 2$ **do** $S = S \cup T$ **end**
return S;

We say that the hypergraph has the *Sylvester property* if at least one of its lines is a two-point set or V.

We often write simply uv for a two-point set $\{u, v\}$, and uvw for a three-point set $\{u, v, w\}$, and so on.

A *Steiner triple system* is a 3-uniform hypergraph $(V, \mathcal{H})$ such that every two-point subset of V is contained in precisely one member of $\mathcal{H}$. The lines of a Steiner triple system are precisely its edges, and so every Steiner triple system on more than three vertices lacks the Sylvester property. It is easy to prove that a Steiner triple system on n

vertices exists only if $n \equiv 1$ or 3 mod 6. The unique (up to isomorphism) Steiner triple system on vertices $1, 2, \ldots, 7$ has edges

$$124, 235, 346, 457, 561, 672, 713;$$

this Steiner triple system is also known as the *Fano configuration* or *finite projective plane over* GF(2) and denoted PG(2, 2). The unique (up to isomorphism) Steiner triple system on vertices $1, 2, \ldots, 9$ has edges

$$123, 456, 789,\ \ 147, 258, 369,\ \ 159, 267, 348,\ \ 168, 249, 357;$$

this Steiner triple system is also known as the *finite affine plane over* GF(3) and denoted AG(2, 3). More on Steiner triple systems may be found, for instance, in [35].

Another 3-uniform hypergraph that lacks the Sylvester property has the nine vertices $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, d_3$ and fourteen edges, namely, the five edges

$$a_1b_1c_1,\ a_1b_2c_2,\ a_2b_1c_2,\ a_2b_2c_1,\ d_1d_2d_3 \tag{2}$$

and the nine edges $a_1a_2d_k,\ b_1b_2d_k,\ c_1c_2d_k$ with $k = 1, 2, 3$. This hypergraph has eight lines, namely, the five lines (2) of size three and the three lines

$$a_1a_2d_1d_2d_3,\ b_1b_2d_1d_2d_3,\ c_1c_2d_1d_2d_3$$

of size five. We let $H_9^\star$ denote this hypergraph.

Lemma 4.2. *A* 3*-uniform hypergraph with at most nine vertices lacks the Sylvester property if and only if it is isomorphic to one of* PG(2, 2), AG(2, 3), *and* $H_9^\star$.

Proof. Consider an arbitrary 3-uniform hypergraph $(V, \mathcal{H})$ which lacks the Sylvester property. In particular,

(i) every line contains at least three vertices,

and so every two vertices are contained in an edge. It follows that

(ii) no two lines intersect in precisely two vertices

(else each edge containing the two vertices of the intersection would share just these two vertices with one of the lines) and that

(iii) if lines L_1, L_2 intersect, then $|L_2 - L_1| \neq 1$

(if $L_2 - L_1 = \{u\}$ and $v \in L_1 \cap L_2$, then each edge containing u and v would share precisely two vertices with one of L_1 and L_2). Writing n for $|V|$, we claim that

(iv) the union of any two lines contains at most $n - 2$ vertices.

To justify this claim, assume the contrary: the union of lines L_1 and L_2 contains at least $n-1$ vertices. Since $|L_1| < n$ and $|L_2| < n$, there are vertices w_1, w_2 such that $w_1 \notin L_1$ and $w_2 \notin L_2$; since $|L_1| \geq 2$ and $|L_2| \geq 2$, there are vertices v_1, v_2 such that $v_1 \in L_1$ and $v_2 \in L_2$; since each edge containing v_iw_i must avoid sharing precisely two points with L_i, we have $|L_1| \leq n-2$ and $|L_2| \leq n-2$. Now there are vertices u_1, u_2 such that

$u_1 \in L_1 - L_2$, $u_2 \in L_2 - L_1$. For every choice of such u_1 and u_2, each edge containing u_1u_2 must avoid sharing precisely two points with L_1 or L_2, and so it has the form u_1u_2x with $x \notin L_1 \cup L_2$; by assumption, an x with $x \notin L_1 \cup L_2$ is unique. It follows that the line u_1u_2 contains $V - (L_1 \cap L_2)$. Since this line does not equal V, we conclude that there is a vertex y in $L_1 \cap L_2$. However, then each edge containing xy shares precisely two points with one of L_1 or L_2; this contradiction concludes our proof of (iv).

From (iii) and (iv), it follows that

(v) every line contains at most $n - 4$ vertices:

if L_1 is a line, if u and v are vertices such that $u \notin L_1$, $v \in L_1$, and if L_2 is the line uv, then (iii) guarantees that $|L_2 - L_1| \geq 2$ and (iv) guarantees that $|V - (L_1 \cup L_2)| \geq 2$.

Next, we propose to show that

(vi) if all lines but one contain precisely three vertices, then the exceptional line contains at most $(n-1)/2$ vertices, and its size has the parity of n.

To prove (vi), enumerate the elements of V as $v_1, v_2, \ldots, v_n$ so that $v_1v_2 \cdots v_k$ is the exceptional line L_0; note that each pair v_iv_j with $1 \leq i < j \leq n$ and $j > k$ is contained only in lines of size three, and so it is contained in precisely one line. Now let $\mathcal{L}$ denote the set of lines that meet L_0 in at most one point. Since each line in $\mathcal{L}$ contains at least one of the pairs v_iv_j with $k < i < j \leq n$ and at most two of the pairs v_iv_j with $1 \leq i \leq k < j \leq n$, it follows that

$$\frac{k(n-k)}{2} \leq |\mathcal{L}| \leq \binom{n-k}{2},$$

and so $k \leq n - k - 1$. Furthermore, since each line in $\mathcal{L}$ contains an even number of the pairs v_kv_j with $k < j \leq n$, it follows that $n - k$ is even. The proof of (vi) is completed.

So far we have made no use of the assumption that $n \leq 9$; from now on, we rely on it constantly. First, under this assumption, we propose to prove that

(vii) no line has size four.

For this purpose, assume the contrary: some line A has size four. Properties (i) and (vi) guarantee that some other line B has size at least four; property (v) guarantees that the size of B is four or five; next, properties (ii), (iii), and (iv) force $|A| = |B| = 4$, $n = 9$, and $|A \cap B| = 1$. Enumerate the vertices of A as a_1, a_2, a_3, d, enumerate the vertices of B as b_1, b_2, b_3, d, and let c_1, c_2 denote the remaining two vertices. For every choice of i and j such that $1 \leq i, j \leq 3$, an edge containing a_ib_j must avoid sharing precisely two vertices with A or B, and so it has the form $a_ib_jc_k$ for some k. Since there are nine choices of i, j and only two available values of k, one of the two vertices c_k must belong to at least five edges of the form $a_ib_jc_k$; symmetry allows as to assume that $k = 1$. In turn, since there are at least five edges of the form $a_ib_jc_1$ and only three available values of i, one of the three vertices a_i must belong to at least two of these edges of this form; symmetry allows as to assume that $i = 1$. Now, with L standing for line a_1c_1, we have $|L - B| \geq 2$ and $|L \cap B| \geq 2$; this contradicts one of (ii), (iii), or (v), and so the proof of (vii) is completed.

Finally, (i), (v), and (vii) allow us to distinguish between two cases.

Case 1: *Every line has size three*. In this case, (ii) guarantees that $(V, \mathcal{H})$ is a Steiner triple system; hence $(V, \mathcal{H})$ is isomorphic to PG(2, 2) or to AG(2, 3).

Case 2: $n = 9$ *and some line has size five*. Let A denote this line. Properties (vi), (vii), and (v) guarantee that some other line B has size five; next, properties (iv) and (iii) force $|A \cap B| = 3$. Enumerate the vertices of A as a_1, a_2, d_1, d_2, d_3, enumerate the vertices of B as b_1, b_2, d_1, d_2, d_3, and let c_1, c_2 denote the remaining two vertices. For each of the three values of k, an edge containing c_1d_k must avoid sharing precisely two vertices with A or B, and so it has the form $c_1c_2d_k$. Hence the line c_1c_2 contains $c_1c_2d_1d_2d_3$; in turn, property (v) guarantees that it includes no additional points; let C denote this line.

We have just proved that

- $c_1c_2d_1 \in \mathcal{H}$, $c_1c_2d_2 \in \mathcal{H}$, $c_1c_2d_3 \in \mathcal{H}$;

the same argument with the roles of A, B, C permuted shows that

- $a_1a_2d_1 \in \mathcal{H}$, $a_1a_2d_2 \in \mathcal{H}$, $a_1a_2d_3 \in \mathcal{H}$

and that

- $b_1b_2d_1 \in \mathcal{H}$, $b_1b_2d_2 \in \mathcal{H}$, $b_1b_2d_3 \in \mathcal{H}$.

An edge containing d_1d_2 must avoid sharing precisely two vertices with A or B or C, and so

- $d_1d_2d_3 \in \mathcal{H}$.

Observe that the ten edges listed so far are the only edges that share at least two vertices with at least one of A, B, C.

Each additional edge has the form $a_ib_jc_k$; observe that any two such edges share at most one vertex (if, for instance, $a_1b_1c_1 \in \mathcal{H}$ and $a_1b_1c_2 \in \mathcal{H}$, then the line a_1b_1 would contain $a_1b_1c_1c_2d_1d_2d_3$, contradicting (v)), and so each of the twelve pairs a_ib_j, a_ic_j, b_ic_j $(1 \leq i, j \leq 2)$ belongs to precisely one of these additional edges. Symmetry allows us to assume that

- $a_1b_1c_1 \in \mathcal{H}$,

after which

- $a_1b_2c_2 \in \mathcal{H}$, $a_2b_1c_2 \in \mathcal{H}$, $a_2b_2c_1 \in \mathcal{H}$

are forced.

To summarize, $(V, \mathcal{H})$ is isomorphic to $H_9^\star$. □

By a *Steiner extension* of a graph G, we mean a Steiner triple system S such that S and G have the same set of vertices and such that each edge of S contains a unique edge of G.

Lemma 4.3. *If there is a metric space* (V, ρ) *such that* $(V, \mathcal{H}(\mathcal{B}(\rho)))$ *is a Steiner triple system*, *then there is a graph* (V, E) *that has a unique Steiner extension*.

Proof. Write $\mathcal{H} = \mathcal{H}(\mathcal{B}(\rho))$ and let G denote the graph with vertex-set V, where vertices x and z are adjacent if and only if $\rho(x, z) = \rho(x, y) + \rho(y, z)$ for some vertex y. By assumption, $(V, \mathcal{H})$ is a Steiner extension of G; we propose to show that $(V, \mathcal{H})$ is the unique Steiner extension of G. For this purpose, enumerate the edges of G as

$x_i z_i$ and enumerate the edges of $(V, \mathcal{H})$ as $x_i y_i z_i$; given an arbitrary Steiner extension $(V, \mathcal{H}')$ of G, enumerate the edges of $(V, \mathcal{H}')$ as $x_i y_i' z_i$. Since both $(V, \mathcal{H})$ and $(V, \mathcal{H}')$ are Steiner triple systems, we have

$$\sum_i (\rho(x_i, y_i) + \rho(y_i, z_i) + \rho(x_i, z_i)) = \frac{1}{2} \sum_u \sum_v \rho(u, v),$$
$$\sum_i (\rho(x_i, y_i') + \rho(y_i', z_i) + \rho(x_i, z_i)) = \frac{1}{2} \sum_u \sum_v \rho(u, v),$$

and so

$$\begin{aligned} \sum_i \rho(x_i, z_i) &\le \sum_i (\rho(x_i, y_i') + \rho(y_i', z_i)) \\ &= \sum_i (\rho(x_i, y_i) + \rho(y_i, z_i)) = \sum_i \rho(x_i, z_i), \end{aligned}$$

and so $\rho(x_i, z_i) = \rho(x_i, y_i') + \rho(y_i', z_i)$ for all i. However, then $y_i' = y_i$ for all i, and so $\mathcal{H}' = \mathcal{H}$. □

Lemma 4.4. *No graph* (V, E) *such that* $1 < |V| < 10$ *has a unique Steiner extension.*

Proof. Computer verification. In fact, every graph (V, E) such that $1 < |V| < 10$ has an even number of Steiner extensions. □

Proof of Theorem 4.1. Consider an arbitrary metric space (V, ρ) such that $1 < |V| < 10$. Lemmas 4.3 and 4.4 together guarantee that

(i) $(V, \mathcal{H}(\mathcal{B}(\rho)))$ is not isomorphic to PG(2, 2)

and that

(ii) $(V, \mathcal{H}(\mathcal{B}(\rho)))$ is not isomorphic to AG(2, 3).

As pointed out by Bruce Reed, (i) implies that

(iii) $(V, \mathcal{H}(\mathcal{B}(\rho)))$ is not isomorphic to $H_9^\star$.

More precisely, write

$$\begin{aligned} W &= \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, d_3\}, \\ W_0 &= \{a_1, a_2, b_1, b_2, c_1, c_2, d_1\} \end{aligned}$$

and note that the edges of $H_9^\star$ contained in W_0 are

$$a_1 b_1 c_1,\ b_1 c_2 a_2,\ c_2 c_1 d_1,\ c_1 a_2 b_2,\ a_2 d_1 a_1,\ d_1 b_2 b_1,\ b_2 a_1 c_2;$$

if there were a metric ρ such that $(W, \mathcal{H}(\mathcal{B}(\rho))) = H_9^\star$, then we would have $(W_0, \mathcal{H}(\mathcal{B}(\rho))) = \text{PG}(2, 2)$, contradicting (i). The conclusion of the theorem follows from (i)–(iii) and Lemma 4.2. □

It is conceivable that Lemma 4.4 generalizes as follows:

Conjecture 4.5. *No finite graph with more than three vertices has a unique Steiner extension.*

In fact, an even stronger statement might hold true:

Conjecture 4.6. *Every finite graph with more than three vertices has an even number of Steiner extensions.*

5. Metric Spaces Induced by Graphs

Every finite connected graph (V, E) induces a metric ρ on V by letting $\rho(a, b)$ be the smallest number of edges in a path from a to b. As in Conjecture 3.2, each of these metric spaces includes distinct points a, b such that the line ab is $\{a, b\}$ or V; in fact, it has an even stronger property.

Theorem 5.1. *In every metric space (V, ρ) induced by a finite connected graph, every line consists either of precisely two points or of the entire V.*

Proof. Two vertices are called *twins* if no third vertex is adjacent to precisely one of them. If a and b are adjacent twins, then $\rho(a, b) = 1$ and all the other vertices c have $\rho(c, a) = \rho(c, b)$; it follows that no member of $\mathcal{H}(\mathcal{B}(\rho))$ includes both a and b, and so line ab consists of a and b alone. We are going to prove that in all other cases, line ab consists of the entire V.

For this purpose, let S denote the line ab. Note that

(i) S includes two nonadjacent vertices:

if a and b are nonadjacent, then (i) holds trivially; if a and b are adjacent, then some vertex c is adjacent to precisely one of them, and so $\{a, b, c\} \in \mathcal{H}(\mathcal{B}(\rho))$, and so $c \in S$. Note also that

(ii) S induces a connected graph:

with any two nonadjacent vertices, S includes all the vertices of every shortest path between these two vertices. Facts (i) and (ii) imply that

(iii) if a vertex has a neighbor in S, then it belongs to S.

To see this, consider an arbitrary vertex u that has a neighbor in S. If u is adjacent to all the vertices of S, then (i) guarantees the existence of vertices v, w in S such that $\rho(u, v) = 1$, $\rho(u, w) = 1$, $\rho(v, w) = 2$; if u is not adjacent to all the vertices of S, then (ii) guarantees the existence of vertices v, w in S such that $\rho(v, w) = 1$, $\rho(u, v) = 1$, $\rho(u, w) = 2$. In either case, $\{u, v, w\} \in \mathcal{H}(\mathcal{B}(\rho))$, and so $u \in S$.

Finally, (iii) and the fact that the graph is connected imply $S = V$. ☐

6. Metric Betweenness

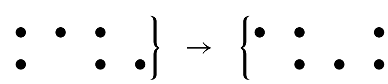
A ternary relation $\mathcal{B}$ on a set V is called a *metric betweenness* if there is a metric ρ on V such that

$$(a, b, c) \in \mathcal{B} \quad \Leftrightarrow \quad a, b, c \text{ are all distinct and } \rho(a, b) + \rho(b, c) = \rho(a, c).$$

Menger [25] seems to have been the first to study this relation. He observed that every metric betweenness $\mathcal{B}$ has properties

(M1) if $(a, b, c) \in \mathcal{B}$, then a, b, c are three distinct points,
(M2) if $(a, b, c) \in \mathcal{B}$, then $(c, a, b) \notin \mathcal{B}$,
(M3) if $(a, b, c) \in \mathcal{B}$, then $(c, b, a) \in \mathcal{B}$,
(M4) if $(a, b, c),\ (a, c, d) \in \mathcal{B}$, then $(a, b, d),\ (b, c, d) \in \mathcal{B}$

and illustrated (M4) by the diagram



whose symmetry makes it clear that (M4) implies its own converse,

$$\text{if } (a, b, d),\ (b, c, d) \in \mathcal{B}, \text{ then } (a, b, c),\ (a, c, d) \in \mathcal{B}.$$

To see that every metric betweenness $\mathcal{B}$ must satisfy (M4), consider an arbitrary metric ρ and let $\mathcal{B}$ denote the betweenness induced by ρ. If $(a, b, c),\ (a, c, d) \in \mathcal{B}$, then

$$\begin{aligned}
\rho(a, c) - \rho(a, b) - \rho(b, c) &= 0,\\
\rho(a, d) - \rho(a, c) - \rho(c, d) &= 0,\\
\rho(a, b) + \rho(b, d) - \rho(a, d) &\geq 0,\\
\rho(b, c) + \rho(c, d) - \rho(b, d) &\geq 0;
\end{aligned}$$

the sum of these four constraints reads $0 \geq 0$, and so ρ must satisfy the two inequalities as equations.

Property (M4), sometimes referred to as "transitivity" of metric betweenness, is pointed out often; two instances are Blumenthal [4, Theorem 12.1] and Busemann [8, (6.6)]. However, attempts to characterize metric betweenness seem to have stopped there; the less obvious observations presented in the remainder of this section may well be new.

Trivially, a ternary relation on a three-point set has properties (M1)–(M3) if and only if it is isomorphic to one of $\emptyset$ and $\{(1, 2, 3), (3, 2, 1)\}$; trivially, either of these two ternary relations is a metric betweenness. Hence properties (M1)–(M3) characterize metric betweenness on three points.

Properties (M1)–(M4) characterize metric betweenness on four points: a ternary relation on a four-point set has these four properties if and only if it is isomorphic to one of

$$\begin{array}{l}
\emptyset,\\
\{(1,2,3),(3,2,1)\},\\
\{(1,2,3),(3,2,1),\ (1,2,4),(4,2,1)\},\\
\{(1,2,3),(3,2,1),\ (2,3,4),(4,3,2)\},\\
\{(1,2,4),(4,2,1),\ (1,3,4),(4,3,1)\},\\
\{(1,4,2),(2,4,1),\ (1,4,3),(3,4,1),\ (2,4,3),(3,4,2)\},\\
\{(1,2,3),(3,2,1),\ (2,3,4),(4,3,2),\ (3,4,1),(1,4,3)\},\\
\{(1,2,3),(3,2,1),\ (1,2,4),(4,2,1),\ (1,3,4),(4,3,1),\ (2,3,4),(4,3,2)\},\\
\{(1,2,3),(3,2,1),\ (2,3,4),(4,3,2),\ (3,4,1),(1,4,3),\ (4,1,2),(2,1,4)\};
\end{array}$$

each of these nine ternary relations is a metric betweenness.

No additional properties are required to characterize metric betweenness on five points: precisely 122 isomorphism types of ternary relations on a five-point set have properties (M1)–(M4) and each of these 122 ternary relations is a metric betweenness.

However, when it comes to characterizing metric betweenness on six points, at least two additional properties are required:

$$\begin{array}{ll}
\text{(M5)} & (b_1,a_1,c_1),\ (b_2,a_1,c_2),\ (b_1,a_2,c_2),\ (b_2,a_2,c_1)\in\mathcal{B}\text{ implies}\\
& (b_1,a_2,c_1),\ (b_2,a_2,c_2),\ (b_1,a_1,c_2),\ (b_2,a_1,c_1)\in\mathcal{B},\\
\text{(M6)} & (b_1,a_1,c_1),\ (a_1,b_2,c_2),\ (a_2,b_1,c_2),\ (b_2,a_2,c_1)\in\mathcal{B}\text{ implies}\\
& (b_1,a_2,c_1),\ (a_1,b_1,c_2),\ (a_2,b_2,c_2),\ (b_2,a_1,c_1)\in\mathcal{B}.
\end{array}$$

To see that every metric betweenness $\mathcal{B}$ must satisfy (M5) and (M6), consider an arbitrary metric ρ and let $\mathcal{B}$ denote the betweenness induced by ρ. If

$$(b_1,a_1,c_1),\ (b_2,a_1,c_2),\ (b_1,a_2,c_2),\ (b_2,a_2,c_1)\in\mathcal{B},$$

then

$$\begin{aligned}
\rho(b_1,c_1)-\rho(b_1,a_1)-\rho(a_1,c_1) &= 0,\\
\rho(b_2,c_2)-\rho(b_2,a_1)-\rho(a_1,c_2) &= 0,\\
\rho(b_1,c_2)-\rho(b_1,a_2)-\rho(a_2,c_2) &= 0,\\
\rho(b_2,c_1)-\rho(b_2,a_2)-\rho(a_2,c_1) &= 0,\\
\rho(b_1,a_2)+\rho(a_2,c_1)-\rho(b_1,c_1) &\ge 0,\\
\rho(b_2,a_2)+\rho(a_2,c_2)-\rho(b_2,c_2) &\ge 0,\\
\rho(b_1,a_1)+\rho(a_1,c_2)-\rho(b_1,c_2) &\ge 0,\\
\rho(b_2,a_1)+\rho(a_1,c_1)-\rho(b_2,c_1) &\ge 0;
\end{aligned}$$

the sum of these eight constraints reads $0 \ge 0$, and so ρ must satisfy the four inequalities as equations. Similarly, if

$$(b_1,a_1,c_1),\ (a_1,b_2,c_2),\ (a_2,b_1,c_2),\ (b_2,a_2,c_1)\in\mathcal{B},$$

then

$$\begin{aligned}
\rho(b_1, c_1) - \rho(b_1, a_1) - \rho(a_1, c_1) &= 0,\\
\rho(a_1, c_2) - \rho(a_1, b_2) - \rho(b_2, c_2) &= 0,\\
\rho(a_2, c_2) - \rho(a_2, b_1) - \rho(b_1, c_2) &= 0,\\
\rho(b_2, c_1) - \rho(b_2, a_2) - \rho(a_2, c_1) &= 0,\\
\rho(b_1, a_2) + \rho(a_2, c_1) - \rho(b_1, c_1) &\geq 0,\\
\rho(a_1, b_1) + \rho(b_1, c_2) - \rho(a_1, c_2) &\geq 0,\\
\rho(a_2, b_2) + \rho(b_2, c_2) - \rho(a_2, c_2) &\geq 0,\\
\rho(b_2, a_1) + \rho(a_1, c_1) - \rho(b_2, c_1) &\geq 0;
\end{aligned}$$

the sum of these eight constraints reads $0 \geq 0$, and so ρ must satisfy the four inequalities as equations.

Properties (M5) and (M6) are related to the notion of a *complete quadrilateral* [11, Section 14.1], which is a configuration of six *points*,

$$a_1, a_2, b_1, b_2, c_1, c_2,$$

and four *lines*,

$$\{a_1, b_1, c_1\},\ \{a_1, b_2, c_2\},\ \{a_2, b_1, c_2\},\ \{a_2, b_2, c_1\}.$$

To elaborate on the relationship, we say that a ternary relation $\mathcal{B}$ is a *betweenness* if it has properties (M1)–(M3); for each betweenness $\mathcal{B}$, we define the *lines* of $\mathcal{B}$ as we have done for $\mathcal{H}(\mathcal{B})$. Now there are 27 distinct betweenness relations on $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ whose lines with more than two points form the complete quadrilateral; each of these 27 betweenness relations is isomorphic to one of the following six betweenness relations:

$\mathcal{B}_1$ consists of
(b_1, a_1, c_1), (b_2, a_1, c_2), (b_1, a_2, c_2), (b_2, a_2, c_1), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, a_2, b_1), (c_1, a_2, b_2).

$\mathcal{B}_2$ consists of
(b_1, a_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2), (b_2, a_2, c_1), and
(c_1, a_1, b_1), (c_2, b_2, a_1), (c_2, b_1, a_2), (c_1, a_2, b_2).

$\mathcal{B}_3$ consists of
(b_1, a_1, c_1), (b_2, a_1, c_2), (a_2, b_1, c_2), (b_2, a_2, c_1), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, b_1, a_2), (c_1, a_2, b_2).

$\mathcal{B}_4$ consists of
(b_1, a_1, c_1), (b_2, a_1, c_2), (a_2, b_1, c_2), (a_2, c_1, b_2), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, b_1, a_2), (b_2, c_1, a_2).

$\mathcal{B}_5$ consists of
(a_1, b_1, c_1), (b_2, a_1, c_2), (b_1, a_2, c_2), (a_2, c_1, b_2), and
(c_1, b_1, a_1), (c_2, a_1, b_2), (c_2, a_2, b_1), (b_2, c_1, a_2).

$\mathcal{B}_6$ consists of
(b_1, a_1, c_1), (b_2, a_1, c_2), (a_2, b_1, c_2), (a_2, b_2, c_1), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, b_1, a_2), (c_1, b_2, a_2).

$\mathcal{B}_1$ is not a metric betweenness, since it lacks property (M5); $\mathcal{B}_2$ is not a metric betweenness, since it lacks property (M6).

Each of the remaining four ternary relations, $\mathcal{B}_3$–$\mathcal{B}_6$, is a metric betweenness. $\mathcal{B}_3$ is induced by the metric specified in the table

	a_1	a_2	b_1	b_2	c_1	c_2
a_1	0	4	2	3	4	3
a_2	4	0	3	2	4	6
b_1	2	3	0	4	6	3
b_2	3	2	4	0	6	6
c_1	4	4	6	6	0	4
c_2	3	6	3	6	4	0

$\mathcal{B}_4$ is induced by the metric specified in the table

	a_1	a_2	b_1	b_2	c_1	c_2
a_1	0	4	2	2	2	2
a_2	4	0	3	5	3	5
b_1	2	3	0	3	4	2
b_2	2	5	3	0	2	4
c_1	2	3	4	2	0	3
c_2	2	5	2	4	3	0

$\mathcal{B}_5$ is induced by the metric specified in the table

	a_1	a_2	b_1	b_2	c_1	c_2
a_1	0	4	3	3	5	3
a_2	4	0	2	6	2	2
b_1	3	2	0	5	2	4
b_2	3	6	5	0	4	6
c_1	5	2	2	4	0	3
c_2	3	2	4	6	3	0

and $\mathcal{B}_6$ is induced by the Euclidean distances between points

$$a_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

To view properties (M3)–(M6) in a more general perspective, we begin with a finite set V and let $\mathcal{B}^\star$ denote the set of all ordered triples of distinct points of V. With each (a, b, c) in $\mathcal{B}^\star$, we associate a vector $v(a, b, c)$ whose components are indexed by two-point subsets of V: this vector is defined by

$$v(a,b,c)_S = \begin{cases} 1 & \text{if} \quad S = \{a,b\} \quad \text{or} \quad S = \{b,c\}, \\ -1 & \text{if} \quad S = \{a,c\}, \\ 0 & \text{otherwise.} \end{cases}$$

Every solution λ of the system

$$\sum(\lambda(a,b,c)v(a,b,c) : (a,b,c) \in \mathcal{B}^\star) = 0$$

defines a pair $(\mathcal{B}^+, \mathcal{B}^-)$ of subsets of $\mathcal{B}^\star$ by

$$\begin{aligned}\mathcal{B}^+ &= \{(a,b,c) \in \mathcal{B}^\star : \lambda(a,b,c) > 0\},\\ \mathcal{B}^- &= \{(a,b,c) \in \mathcal{B}^\star : \lambda(a,b,c) < 0\};\end{aligned}$$

let $\mathcal{F}(V)$ denote the set of all these pairs $(\mathcal{B}^+, \mathcal{B}^-)$. (The passage from $\{v(a,b,c) : (a,b,c) \in \mathcal{B}^\star\}$ to $\mathcal{F}(V)$ generalizes in the theory of *oriented matroids*: see, for instance, [36].)

If $(\mathcal{B}^+, \mathcal{B}^-) \in \mathcal{F}(V)$, then every metric betweenness on V has property

(M*) $\mathcal{B}^- \subseteq \mathcal{B}$ implies $\mathcal{B}^+ \subseteq \mathcal{B}$.

To see this, consider an arbitrary metric ρ and let $\mathcal{B}$ denote the betweenness induced by ρ. If $\mathcal{B}^- \subseteq \mathcal{B}$, then

$$\begin{aligned}v(a,b,c)^T\rho &= 0 \qquad \text{for all } (a,b,c) \text{ in } \mathcal{B}^-,\\ v(a,b,c)^T\rho &\geq 0 \qquad \text{for all } (a,b,c) \text{ in } \mathcal{B}^+;\end{aligned}$$

the weighted sum of these constraints reads $0 \geq 0$, and so ρ must satisfy the inequalities as equations.

Conditions (M3)–(M6) are special cases of (M*); six additional special cases of (M*) are as follows:

(M7) $(a,b,c),\ (d,b,e),\ (e,c,f),\ (a,e,g),\ (c,g,d),\ (f,b,g) \in \mathcal{B}$ implies $(a,b,g),\ (a,e,c),\ (c,b,d),\ (d,g,e),\ (e,b,f),\ (f,c,g) \in \mathcal{B}$,

(M8) $(a,b,c),\ (d,b,e),\ (a,f,e),\ (c,d,f),\ (a,d,g),\ (b,f,g) \in \mathcal{B}$ implies $(a,d,c),\ (a,b,e),\ (c,b,f),\ (d,f,e),\ (a,f,g),\ (b,d,g) \in \mathcal{B}$,

(M9) $(a,b,c),\ (d,b,e),\ (a,f,e),\ (c,d,f),\ (f,b,g),\ (d,a,g) \in \mathcal{B}$ implies $(a,b,e),\ (a,d,c),\ (c,b,f),\ (d,b,g),\ (d,f,e),\ (f,a,g) \in \mathcal{B}$,

(M10) $(a,b,c),\ (c,d,e),\ (e,a,f),\ (b,e,g),\ (c,f,g),\ (a,g,d) \in \mathcal{B}$ implies $(a,e,d),\ (a,f,c),\ (b,a,g),\ (c,b,e),\ (c,d,g),\ (e,g,f) \in \mathcal{B}$,

(M11) $(a,b,c), (d,b,e), (a,f,e), (c,d,f), (a,g,d), (b,f,g), (e,c,g) \in \mathcal{B}$ implies $(a,b,e), (a,f,d), (a,g,c), (b,d,g), (c,b,f), (d,c,e), (e,f,g) \in \mathcal{B}$,

(M12) $(a,b,c), (b,d,e), (a,f,e), (d,c,f), (c,e,g), (d,a,g), (b,g,f) \in \mathcal{B}$ implies $(a,d,c), (b,c,e), (b,a,f), (d,e,f), (a,g,e), (c,f,g), (d,b,g) \in \mathcal{B}$.

Trivially, if $\mathcal{B}$ is a metric betweenness on V, then $\mathcal{B}$ satisfies (M1), (M2) and there is a solution ρ of the system

$$\begin{aligned}v(a,b,c)^T\rho &= 0 \qquad \text{for all } (a,b,c) \text{ in } \mathcal{B},\\ v(a,b,c)^T\rho &> 0 \qquad \text{for all } (a,b,c) \text{ in } \mathcal{B}^\star - \mathcal{B}.\end{aligned} \tag{3}$$

Conversely, if a ternary relation $\mathcal{B}$ on V satisfies (M1), (M2) and if the system (3) is solvable, then $\mathcal{B}$ is a metric betweenness on V. More precisely, if a ternary relation $\mathcal{B}$ on V satisfies (M2) then, for every choice of distinct points a, b, c of V, the sum of the two constraints in (3) that arise from (a, b, c) and (b, a, c) reads $2\rho_{\{a,b\}} > 0$, and so every solution of (3) defines a metric on V; if $\mathcal{B}$ satisfies (M1), then it is induced by this metric.

The duality principle of linear programming, stated in terms of Kuhn [22], guarantees that system (3) is unsolvable if and only if it is *inconsistent* in the following sense: one can assign a weight to each equation in (3) and a nonnegative weight to each inequality in (3) in such a way that the resulting weighted sum of all these constraints reads $0 > 0$. Note that (3) is inconsistent if and only if it violates at least one of the conditions ($\mathrm{M}^\star$) with $(\mathcal{B}^+, \mathcal{B}^-)$ in $\mathcal{F}(V)$. Hence a ternary relation $\mathcal{B}$ on V is a metric betweenness if and only if it satisfies (M1), (M2), and all the conditions ($\mathrm{M}^\star$) with $(\mathcal{B}^+, \mathcal{B}^-)$ ranging through $\mathcal{F}(V)$. It is doubtful whether, when V is large, these conditions can be described in a way that is both explicit and concise.

The fundamental result of Khachiyan [20] guarantees that solvability of (3) can be tested in polynomial time; it follows that metric betweenness can be recognized in polynomial time.

7. ℓ_1-Betweenness

We say that a metric betweenness on a set V is an ℓ_p*-betweenness* if $\mathcal{B}$ is induced by an ℓ_p-metric on V. We present an easy characterization of ℓ_1-betweenness in terms of the following notion: a *convex subset* of V is any subset S of V such that

$$\text{if } (a, b, c) \in \mathcal{B} \text{ and if } a, c \in S, \text{ then } b \in S.$$

Theorem 7.1. *A ternary relation $\mathcal{B}$ on a finite set V is an ℓ_1-betweenness if and only if it has the following three properties*:

- (M1) *if $(a, b, c) \in \mathcal{B}$, then a, b, c are three distinct points*;
- (M2) *if $(a, b, c) \in \mathcal{B}$, then $(c, a, b) \notin \mathcal{B}$*;
- (L1) *if a, b, c are three distinct points of V such that $(a, b, c) \notin \mathcal{B}$, then there are convex subsets S, T of V such that $S \cap T = \emptyset$, $S \cup T = V$, and $a, c \in S$, $b \in T$.*

Proof. (The "if" part) If $|V| \leq 2$, then assumption (M1) guarantees that $\mathcal{B} = \emptyset$, and so $\mathcal{B}$ is an ℓ_1-betweenness . Now we assume that $|V| \leq 3$. By assumption (L1), there are convex subsets $S_1, T_1, S_2, T_2, \ldots, S_m, T_m$ of V such that $S_i \cap T_i = \emptyset$, $S_i \cup T_i = V$ for all $i = 1, 2, \ldots, m$ and such that, for every choice of distinct points a, b, c of V with $(a, b, c) \notin \mathcal{B}$, there is a subscript i with $a, c \in S_i$, $b \in T_i$. Assumption (M2) guarantees that $m > 0$ and that the mapping φ: $V \to \{0, 1\}^m$ defined by $\varphi(x) = (x_1, x_2, \ldots, x_m)$ with

$$x_i = \begin{cases} 0 & \text{if} \quad x \in S_i\,, \\ 1 & \text{if} \quad x \in T_i \end{cases}$$

is one-to-one. Define ρ by $\rho(x, y) = ||\varphi(x) - \varphi(y)||_1$ and consider an arbitrary choice of distinct points a, b, c of V. If $(a, b, c) \in \mathcal{B}$, then, as each of $S_1, T_1, S_2, T_2, \dots, S_m, T_m$ is convex, we have $\min\{a_i, c_i\} \le b_i \le \max\{a_i, c_i\}$ for all i, and so $\rho(a, b) + \rho(b, c) = \rho(a, c)$; if $(a, b, c) \notin \mathcal{B}$, then there is a subscript i with $a_i = c_i$, $a_i \neq b_i$, and so $\rho(a, b) + \rho(b, c) > \rho(a, c)$. These observations along with assumption (M1) imply that $\mathcal{B}$ is induced by ρ.

(The "only if" part) By assumption, there is a one-to-one mapping $\varphi\colon V \to \mathbf{R}^m$ such that the metric ρ defined by $\rho(x, y) = ||\varphi(x) - \varphi(y)||_1$ induces $\mathcal{B}$. Since $\mathcal{B}$ is a metric betweenness, it has properties (M1) and (M2). To see that $\mathcal{B}$ has property (L1), consider an arbitrary choice of distinct points a, b, c of V and write

$$\varphi(a) = (a_1, a_2, \dots, a_m), \qquad \varphi(b) = (b_1, b_2, \dots, b_m), \qquad \varphi(c) = (c_1, c_2, \dots, c_m).$$

If $\rho(a, b) + \rho(b, c) > \rho(a, c)$, then there is a subscript i with $b_i < \min\{a_i, c_i\}$ or $\max\{a_i, c_i\} < b_i$; in the former case, set

$$S^\star = \{(x_1, x_2, \dots, x_m) \in \mathbf{R}^m : x_i > b_i\},$$
$$T^\star = \{(x_1, x_2, \dots, x_m) \in \mathbf{R}^m : x_i \le b_i\};$$

in the latter case, set

$$S^\star = \{(x_1, x_2, \dots, x_m) \in \mathbf{R}^m : x_i < b_i\},$$
$$T^\star = \{(x_1, x_2, \dots, x_m) \in \mathbf{R}^m : x_i \ge b_i\};$$

in either case, $\{v \in V : \varphi(v) \in S^\star\}$ and $\{v \in V : \varphi(v) \in T^\star\}$ have the properties required of S and T in (L1). □

Corollary 7.2. *Every ℓ_2-betweenness on a finite set is an ℓ_1-betweenness.*

Incidentally, a theorem stronger than Corollary 7.2 asserts that

$$\textit{every } \ell_2\textit{-metric is an } \ell_1\textit{-metric}. \tag{4}$$

This follows from a result of Bretagnolle et al. [7, Theorem 2, p. 238] combined with a result of Schoenberg [30, Corollary 1, p. 527 or Theorem 5, p. 534]. The proofs in these papers are rather sophisticated; a relatively simple proof of (4) was given by Critchley and Fichet [12]. A wealth of additional information on ℓ_1-metrics can be found in the monograph by Deza and Laurent [14].

The notion of ℓ_2-betweenness is more restricted than the notion of ℓ_1-betweenness. For instance, if

$$a = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then the betweenness induced on $\{a, b, c, d\}$ by the ℓ_1-metric is

$$\{(a, b, c), (c, b, a),\ (b, c, d), (d, c, b),\ (c, d, a), (a, d, c),\ (d, a, b), (b, a, d)\};$$

and so it lacks the property

$$\text{if } (a,b,c) \in \mathcal{B} \text{ and } (b,c,d) \in \mathcal{B}, \text{ then } (a,b,d) \in \mathcal{B},$$

which every ℓ_2-betweenness has.

In turn, the notion of ℓ_1-betweenness is more restricted than the notion of metric betweenness. For instance, consider the set $\{v_1, v_2, v_3, w_1, w_2\}$ and the ternary relation $\mathcal{B}$ on this set that consists of the twelve ordered triples (v_i, w_j, v_k) with $i \neq k$ and the six ordered triples (w_i, v_j, w_k) with $i \neq k$. This $\mathcal{B}$ is a metric betweenness: it is induced by the metric ρ with $\rho(v_i, w_j) = 1$ for all i, j and with $\rho(x, y) = 2$ for all other choices of distinct x and y. However, this $\mathcal{B}$ is not an ℓ_1-betweenness as it lacks property (L1) of Theorem 7.1: there are no disjoint convex sets S, T such that $S \cup T = \{v_1, v_2, v_3, w_1, w_2\}$, and $v_1, v_2 \in S$, $v_3 \in T$.

In this sense, the following conjecture is weaker than Conjecture 3.2, but its validity would still strengthen the Sylvester–Gallai theorem.

Conjecture 7.3. *If (V, ρ) is a metric space such that $1 < |V| < \infty$ and ρ is an ℓ_1-metric, then V includes distinct points a, b such that the line ab is $\{a, b\}$ or V.*

In comparing Conjecture 7.3 with the Sylvester–Gallai theorem, it may be interesting to note that

> *a metric betweenness on six points, whose lines with more than two points form the complete quadrilateral, is an ℓ_1-betweenness if and only if it is an ℓ_2-betweenness.*

More precisely, there are four isomorphism types of metric betweenness on six points whose lines with more than two points form the complete quadrilateral; these have been identified as $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_5$, $\mathcal{B}_6$ in Section 6. In that section we noted that $\mathcal{B}_6$ is an ℓ_2 betweenness; it follows that $\mathcal{B}_6$ is an ℓ_1-betweenness; now we are going to point out that none of $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_5$ is an ℓ_1 betweenness. $\mathcal{B}_3$ consists of

(b_1, a_1, c_1), (b_2, a_1, c_2), (a_2, b_1, c_2), (b_2, a_2, c_1), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, b_1, a_2), (c_1, a_2, b_2);

there are no disjoint convex sets S, T such that $S \cup T = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ and $a_2, c_2 \in S$, $a_1 \in T$. $\mathcal{B}_4$ consists of

(b_1, a_1, c_1), (b_2, a_1, c_2), (a_2, b_1, c_2), (a_2, c_1, b_2), and
(c_1, a_1, b_1), (c_2, a_1, b_2), (c_2, b_1, a_2), (b_2, c_1, a_2);

there are no disjoint convex sets S, T such that $S \cup T = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ and $b_1, b_2 \in S$, $a_1 \in T$. $\mathcal{B}_5$ consists of

(a_1, b_1, c_1), (b_2, a_1, c_2), (b_1, a_2, c_2), (a_2, c_1, b_2), and
(c_1, b_1, a_1), (c_2, a_1, b_2), (c_2, a_2, b_1), (b_2, c_1, a_2);

there are no disjoint convex sets S, T such that $S \cup T = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ and $c_1, c_2 \in S$, $a_2 \in T$.

Chen [9] proved the following special case of Conjecture 7.3: *If* (V, ρ) *is a metric space such that* $1 < |V| < \infty$, $V \subset \mathbf{R}^2$, *and* $\rho(x, y) = ||x - y||_1$, *then* V *includes distinct points* a, b *such that the line* ab *is* $\{a, b\}$ *or* V.

8. ℓ_∞-Betweenness

The ℓ_∞-*norm* on $\mathbf{R}^m$ is defined by

$$||(z_1, z_2, \ldots, z_m)||_\infty = \max\{|z_i| : i = 1, 2, \ldots, m\};$$

its name comes from the fact that

$$||z||_\infty = \lim_{p\to\infty} ||z||_p \qquad \text{for all } z;$$

if $\varphi\colon\ V \to \mathbf{R}^m$ is a one-to-one mapping, then the mapping $\rho\colon\ (V \times V) \to \mathbf{R}$ defined by

$$\rho(x, y) = ||\varphi(x) - \varphi(y)||_\infty$$

is a metric and it is called an ℓ_∞-*metric*. Restricting ρ in Conjecture 3.2 to ℓ_∞-metrics (just as Conjecture 7.3 restricts it to ℓ_1-metrics) does not make the conjecture any easier:

$$\textit{if } (V, \rho) \textit{ is a finite metric space, then } \rho \textit{ is an } \ell_\infty\textit{-metric.} \tag{5}$$

To prove (5), enumerate the elements of V as $v_1, v_2, \ldots, v_m$ and define $\varphi\colon\ V \to \mathbf{R}^m$ by

$$\varphi(v) = (\rho(v, v_1), \rho(v, v_2), \ldots, \rho(v, v_m));$$

it is an easy exercise to verify that $\rho(x, y) = ||\varphi(x) - \varphi(y)||_\infty$ for all x and y. (A theorem of Fréchet [17] related to (5) states that every separable metric space is isometrically embeddable in the space of all bounded sequences of real numbers endowed with the supremum norm.)

Victor Klee suggested that, in working on Conjecture 3.2, some researchers might prefer to think in terms of a concrete representation of the metric, such as the universal representation of finite metric spaces (V, ρ) provided by fact (5): we may take $V \subset \mathbf{R}^m$ for some positive integer m and $\rho(x, y) = ||x - y||_\infty$ for all x and y in V.

9. Strict Order Betweenness

Following Lihová [23], we say that a ternary relation $\mathcal{B}$ on a set V is a *strict order betweenness* if there is a partial order $\prec$ on V such that $(a, b, c) \in \mathcal{B}$ if and only if

$$a \prec b \prec c \quad \text{or} \quad c \prec b \prec a.$$

(Strict order betweenness is a variation on the theme of *order betweenness*, where $\prec$ is replaced by $\preceq$; order betweenness was introduced by Birkhoff [3] and characterized, in two different ways, by Altwegg [1] and by Sholander [31].)

Theorem 9.1. *Every strict order betweenness on a finite set is an ℓ_1-betweenness.*

Proof. Consider an arbitrary finite set V and an arbitrary partial order $\prec$ on V; let $\mathcal{B}$ denote the corresponding order betweenness. Given an arbitrary element b of V, set

$$T' = \{x : x \preceq b\}, \qquad S' = V - T',$$
$$T'' = \{x : b \preceq x\}, \qquad S'' = V - T''.$$

All four of these sets, S', T', S'', T'', are convex with respect to $\mathcal{B}$; furthermore, for every choice of points a, c of V such that a, b, c are all distinct and $(a, b, c) \notin \mathcal{B}$, we have $a, c \in S'$ or $a, c \in S''$. Hence $\mathcal{B}$ has property (L1) of Theorem 7.1. □

An example of a strict order betweenness that is not an ℓ_2-betweenness is

$$\{(1, 2, 3), (3, 2, 1),\ (1, 2, 4), (4, 2, 1)\};$$

an example of an ℓ_2-betweenness that is not a strict order betweenness is

$$\{(1, 3, 5), (5, 3, 1),\ (2, 3, 4), (4, 3, 2)\}.$$

Theorem 9.2. *If V is a finite set, if $\mathcal{B}$ is a strict order betweenness on V, and if $\mathcal{H} = \{\{a, b, c\} : (a, b, c) \in \mathcal{B}\}$, then the 3-uniform hypergraph $(V, \mathcal{H})$ has the Sylvester property.*

Proof. Consider an arbitrary partial order $\prec$ on V which defines $\mathcal{B}$. If V includes distinct points a, b that are incomparable (in the sense that neither $a \prec b$ nor $b \prec a$), then the line ab defined by $\mathcal{H}$ is $\{a, b\}$; else every line defined by $\mathcal{H}$ is V. □

10. Related Theorems

Victor Klee and Benny Sudakov (independently of each other) suggested looking at additional theorems in geometry that, like the Sylvester–Gallai theorem, can be stated in terms of point-line incidences and possibly could extend to general metric spaces. Two such theorems, both discussed in Section 6 ("Points on Lines") of [21], go as follows.

Theorem 10.1 [13]. *Every noncollinear set of n points in the plane determines at least n lines.*

Theorem 10.2 [2], [32]. *There is a positive ε such that every noncollinear set of n points in the plane includes a point that lies on at least εn lines.*

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Note added in proof. In September 2003 Xiaomin Chen proved Conjecture 3.2.