

Fall 2011

INDU 6111 Theory of Operations Research
Homework Assignment 2
Solutions

1. For each of the three LP problems in Problem 2.1 (page 26), write down the dual problem and give optimal solutions of both the primal and the dual problems.

a.

$$\begin{array}{lll} \text{minimize} & 4y_1 + 5y_2 + 7y_3 \\ \text{subject to} & y_1 + 2y_2 + 2y_3 \geq 3 \\ & y_1 + y_3 \geq 2 \\ & 2y_1 + 3y_2 + 3y_3 \geq 4 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

$$x_1^* = 2.5, x_2^* = 1.5, x_3^* = 0, y_1^* = 2, y_2^* = 0.5, y_3^* = 0$$

b.

$$\begin{array}{lll} \text{minimize} & 5y_1 + 3y_2 \\ \text{subject to} & y_1 + y_2 \geq 5 \\ & 2y_1 + y_2 \geq 6 \\ & 3y_1 + 2y_2 \geq 9 \\ & y_1 + 3y_2 \geq 8 \\ & y_1, y_2 \geq 0 \end{array}$$

$$x_1^* = 1, x_2^* = 2, x_3^* = 0, x_4^* = 0, y_1^* = 1, y_2^* = 4$$

c.

$$\begin{array}{lll} \text{minimize} & 3y_1 + y_2 + 4y_3 + 5y_4 \\ \text{subject to} & 2y_1 + y_2 + 2y_3 + 4y_4 \geq 2 \\ & 3y_1 + 5y_2 + y_3 + y_4 \geq 1 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

$$x_1^* = 1, x_2^* = 0, y_1^* = 0, y_2^* = 2, y_3^* = 0, y_4^* = 0$$

2. Use Theorem 5.3 to answer the following questions:

A. True or false?

$$x_1^* = 1, x_2^* = 0, x_3^* = 1, x_4^* = 0, x_5^* = 2$$

is an optimal solution of the problem

$$\begin{aligned} \text{maximize} \quad & 8x_1 + 8x_2 + 9x_3 + 6x_4 + 5x_5 \\ \text{subject to} \quad & x_1 - x_2 + 2x_3 - x_4 + x_5 \leq 5, \\ & x_1 + x_2 + 3x_3 + 3x_4 + 2x_5 \leq 9, \\ & 3x_1 + x_2 + 2x_3 + 2x_4 + x_5 \leq 8, \\ & 2x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leq 8, \\ & 2x_1 + 3x_2 + 2x_3 + 2x_4 + x_5 \leq 6, \\ & x_1 + x_2 + x_3 + x_4 + x_5 \leq 4, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Solution: True. System (5.22) reads

$$\begin{aligned} y_1 + y_2 + 3y_3 + 2y_4 + 2y_5 + y_6 &= 8 \\ 2y_1 + 3y_2 + 2y_3 + 2y_4 + 2y_5 + y_6 &= 9 \\ y_1 + 2y_2 + y_3 + y_4 + y_5 + y_6 &= 5 \\ y_2 &= 0 \\ y_3 &= 0 \\ y_4 &= 0 \end{aligned}$$

and it has a unique solution,

$$y_1^* = 1, y_2^* = 0, y_3^* = 0, y_4^* = 0, y_5^* = 3, y_6^* = 1.$$

Since $y_1^* \geq 0, y_2^* \geq 0, y_3^* \geq 0, y_4^* \geq 0, y_5^* \geq 0, y_6^* \geq 0$ and

$$\begin{aligned} -y_1^* + y_2^* + y_3^* + 2y_4^* + 3y_5^* + y_6^* &\geq 8, \\ -y_1^* + 3y_2^* + 2y_3^* + y_4^* + 2y_5^* + y_6^* &\geq 6, \end{aligned}$$

this is a feasible (and optimal) solution of the dual problem.

B. True or false?

$$x_1^* = 1, x_2^* = 0, x_3^* = 1, x_4^* = 1, x_5^* = 1, x_6^* = 0$$

is an optimal solution of the problem

$$\begin{array}{llllllllll} \text{maximize} & 8x_1 & + & 4x_2 & + & 6x_3 & + & 4x_4 & + & 3x_5 & + & 4x_6 \\ \text{subject to} & 2x_1 & - & x_2 & + & x_3 & + & x_4 & + & x_5 & + & 2x_6 & \leq 5, \\ & 2x_1 & + & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & & & \leq 8, \\ & 2x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & + & x_5 & & & \leq 6, \\ & 2x_1 & + & x_2 & + & x_3 & + & x_4 & & & + & x_6 & \leq 4, \\ & x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & + & 2x_6 & \leq 8, \\ & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0. \end{array}$$

Solution: False. System (5.22) reads

$$\begin{array}{llllll} 2y_1 & + & 2y_2 & + & 2y_3 & + & 2y_4 & + & y_5 & = & 8 \\ y_1 & + & 2y_2 & + & 2y_3 & + & y_4 & + & y_5 & = & 6 \\ y_1 & + & y_2 & + & y_3 & + & y_4 & + & 2y_5 & = & 4 \\ y_1 & + & 2y_2 & + & y_3 & & & + & 2y_5 & = & 3 \\ & & & & y_2 & & & & & = & 0 \\ & & & & & & & & y_5 & = & 0 \end{array}$$

and it has a unique solution,

$$y_1^* = 1, y_2^* = 0, y_3^* = 2, y_4^* = 1, y_5^* = 0.$$

Since $2y_1^* + y_4^* + 2y_5^* < 4$, this is not a feasible solution of the dual problem.

C. True or false?

$$x_1^* = 0, x_2^* = 1, x_3^* = 0, x_4^* = 1, x_5^* = 3, x_6^* = 0$$

is an optimal solution of the problem

$$\begin{aligned} \text{maximize} \quad & 8x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 8x_6 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 \leq 5, \\ & 3x_1 + x_2 + 2x_3 + x_4 + 2x_5 + 3x_6 \leq 9, \\ & x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 \leq 6, \\ & x_1 + x_2 + 3x_3 + x_4 + 2x_5 + 2x_6 \leq 9, \\ & 3x_1 + x_2 + 2x_3 + 2x_4 + x_5 + 3x_6 \leq 6, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Solution: False. System (5.22) reads

$$\begin{array}{rcl} y_1 + y_2 + 2y_3 + y_4 + y_5 & = & 9 \\ y_1 + y_2 + y_3 + y_4 + 2y_5 & = & 7 \\ y_1 + 2y_2 + y_3 + 2y_4 + y_5 & = & 6 \\ y_2 & & = 0 \\ y_4 & & = 0 \end{array}$$

and it has a unique solution,

$$y_1^* = 2, y_2^* = 0, y_3^* = 3, y_4^* = 0, y_5^* = 1.$$

Since $y_1^* + 2y_2^* + y_3^* + 3y_4^* + 2y_5^* < 8$, this is not a feasible solution of the dual problem.