

INDU 6111 Theory of Operations Research
Homework Assignment 3
Solutions

1. Consider the basic feasible solution

$$x_1^* = 0, x_2^* = 0, x_3^* = 1, x_4^* = 1, x_5^* = 2, x_6^* = 2$$

of the problem

$$\begin{array}{llllllll} \text{maximize} & 3x_1 & + & 6x_2 & + & 6x_3 & + & 7x_4 & + & 7x_5 & + & 10x_6 \\ \text{subject to} & 2x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & + & 4x_5 & + & 4x_6 & = & 22 \\ & 2x_1 & + & 3x_2 & + & 3x_3 & + & 4x_4 & + & 4x_5 & + & 5x_6 & = & 25 \\ & & & & & & & & & 0 \leq x_1, x_2, x_3, x_4, x_5, x_6 \leq 2 \end{array}$$

What are the basic variables and what are all the candidates for entering the basis?

The answer: x_3, x_4 are the basic variables and x_2, x_5 are all the candidates for entering the basis.

How to get it: Since x_3, x_4 are the only variables with values strictly between their lower and upper bounds, they must be basic. Solving the system $y^T \begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix} = [6, 7]$, we find $y^T = [6, 7]$. Comparing $c_N^T = [3, 6, 7, 10]$ with $y^T A_N = [4, 5, 8, 9]$, we see that the objective function would increase by decreasing x_1 or increasing x_2 or decreasing x_5 or increasing x_6 . Since x_1 is at its lower bound, it cannot be decreased; since x_6 is at its upper bound, it cannot be increased.

2. Find a solution of the system

$$\begin{array}{rrrrrr} x_1 & - & 2x_2 & + & 2x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & = & 1 \\ 3x_1 & - & 4x_2 & + & 8x_3 & - & 5x_4 & = & -1 \\ & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

One solution: $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 2$

Another solution: $x_1 = 0, x_2 = 0, x_3 = 3, x_4 = 5$

There are infinitely many solutions, $x_1 = 3a + 3b, x_2 = b + c, x_3 = 3 - 3a + 3c, x_4 = 5 - 3a + b + 4c$ with $0 \leq a \leq 1, b \geq 0, c \geq 0$.

3. Write down the dual of the problem

$$\begin{array}{llllll} \text{maximize} & & & & x_3 & \\ \text{subject to} & x_1 & + & x_2 & & = & 3 \\ & x_1 & - & 3x_2 & - & x_3 & \geq & 0 \\ & 3x_1 & - & 5x_2 & + & x_3 & \leq & 0 \\ & & & & x_1 \geq 0, & x_2 \geq 0 \end{array}$$

and solve both problems.

The answer: The dual is

$$\begin{array}{ll} \text{minimize} & 3y_1 \\ \text{subject to} & y_1 - y_2 + 3y_3 \geq 0 \\ & y_1 + 3y_2 - 5y_3 \geq 0 \\ & y_2 + y_3 = 1 \\ & y_2 \geq 0, y_3 \geq 0 \end{array}$$

the optimal solution of primal is $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, and the optimal solution of dual is $y_1 = -1/3$, $y_2 = 2/3$, $y_3 = 1/3$.

4. Illustrate Theorem 9.4 on the system

$$\begin{array}{rrcr} x & +3y & +z & \leq 4 \\ -x & -y & +3z & \leq -3 \\ -3x & +2y & & \leq 0 \\ & 2y & -z & \leq 1 \\ x & -2y & -z & \leq -1 \end{array}$$

and certify that the smaller system is inconsistent.

One answer: The system remains unsolvable even if the first inequality is removed. One certificate of inconsistency of the smaller system are the multipliers 10 (at the second inequality), 1 (at the third inequality), 17 (at the fourth inequality), 13 (at the fifth inequality).

One way of getting it: We are asked to find a nonnegative solution of the system

$$\begin{array}{rrrrrr} y_1 & -y_2 & -3y_3 & & +y_5 & = & 0 \\ 3y_1 & -y_2 & +2y_3 & +2y_4 & -2y_5 & = & 0 \\ y_1 & +3y_2 & & -y_4 & -y_5 & = & 0 \\ 4y_1 & -3y_2 & & +y_4 & -y_5 & = & -1 \end{array}$$

with at most four of the variables y_1, y_2, y_3, y_4, y_5 having positive values. A routine way of doing this is to solve the problem

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & y_1 - y_2 - 3y_3 + y_5 = 0 \\ & 3y_1 - y_2 + 2y_3 + 2y_4 - 2y_5 = 0 \\ & y_1 + 3y_2 - y_4 - y_5 = 0 \\ & 4y_1 - 3y_2 + y_4 - y_5 - w = -1 \\ & y_1, y_2, y_3, y_4, y_5, w \geq 0 \end{array}$$

by the revised simplex method. The first step is to get a basis matrix. We can get one right away by introducing slack variables s_1, s_2, s_3 for the first three equations, with each s_i constrained by $0 \leq s_i \leq 0$: our initial basic feasible solution is

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The first iteration: Solving the system

$$y^T \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = [0 \ 0 \ 0 \ 1] \text{ we get } y^T = [0 \ 0 \ 0 \ -1]$$

Comparing $c_N^T = [0, 0, 0, 0, 0]$ with $y^T A_N = [-4, 3, 0, -1, 1]$, we see that the candidates for entering the basis are y_2 and y_5 . Let us choose y_2 . Solving the system

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix} d = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \text{ we get } d = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 3 \end{bmatrix}$$

Since the first three components of d are nonzero, the iteration is degenerate and any of the three slack variables may leave the basis. Arbitrarily, we let s_1 leave and discard it from the problem: our next basic feasible solution is

$$\begin{bmatrix} y_2 \\ s_2 \\ s_3 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The second iteration: Solving the system

$$y^T \begin{bmatrix} -1 & & & & \\ -1 & 1 & & & \\ 3 & & 1 & & \\ -3 & & & 1 & \\ & & & & -1 \end{bmatrix} = [0 \ 0 \ 0 \ 1] \text{ we get } y^T = [3 \ 0 \ 0 \ -1]$$

Comparing $c_N^T = [0, 0, 0, 0]$ with $y^T A_N = [-1, -9, -1, 4]$, we see that y_5 is the only candidate for entering the basis. Solving the system

$$\begin{bmatrix} -1 & & & & \\ -1 & 1 & & & \\ 3 & & 1 & & \\ -3 & & & 1 & \\ & & & & -1 \end{bmatrix} d = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \text{ we get } d = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$$

Since the second and the third components of d are nonzero, the iteration is degenerate and either of s_2, s_3 may leave the basis. Arbitrarily, we let s_2 leave and discard it from the problem: our next basic feasible solution is

$$\begin{bmatrix} y_2 \\ y_5 \\ s_3 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The third iteration: Solving the system

$$y^T \begin{bmatrix} -1 & 1 & & & \\ -1 & -2 & & & \\ 3 & -1 & 1 & & \\ -3 & -1 & & 1 & \\ & & & & -1 \end{bmatrix} = [0 \ 0 \ 0 \ 1] \text{ we get } y^T = [5/3 \ 4/3 \ 0 \ -1]$$

Comparing $c_N^T = [0, 0, 0]$ with $y^T A_N = [5/3, -7/3, 5/3]$, we see that y_1 and y_4 are the candidate for entering the basis. Let us choose y_4 . Solving the system

$$\begin{bmatrix} -1 & 1 & & & \\ -1 & -2 & & & \\ 3 & -1 & 1 & & \\ -3 & -1 & & 1 & \\ & & & & -1 \end{bmatrix} d = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \text{ we get } d = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 5/3 \end{bmatrix}$$

Since the third components of d is nonzero, the iteration is degenerate; s_3 must leave the basis and will be discarded from the problem. Our next basic feasible solution is

$$\begin{bmatrix} y_2 \\ y_5 \\ y_4 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The fourth iteration: Solving the system

$$y^T \begin{bmatrix} -1 & 1 \\ -1 & -2 & 2 \\ 3 & -1 & -1 \\ -3 & -1 & 1 & -1 \end{bmatrix} = [0 \quad 0 \quad 0 \quad 1] \quad \text{we get} \quad y^T = [-10 \quad -2 \quad -5 \quad -1]$$

Comparing $c_N^T = [0, 0]$ with $y^T A_N = [-25, 26]$, we see that y_3 is the only candidate for entering the basis. Solving the system

$$\begin{bmatrix} -1 & 1 \\ -1 & -2 & 2 \\ 3 & -1 & -1 \\ -3 & -1 & 1 & -1 \end{bmatrix} d = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \text{we get} \quad d = \begin{bmatrix} -10 \\ -13 \\ -17 \\ 26 \end{bmatrix}$$

As the value of y_3 increases from 0 to a positive level t , the values of the basic variables get adjusted as

$$\begin{bmatrix} y_2 \\ y_5 \\ y_4 \\ w \end{bmatrix} = \begin{bmatrix} 10t \\ 13t \\ 17t \\ 1 - 26t \end{bmatrix}.$$

The largest value of t which keeps the solution feasible is $1/26$; when t reaches this level, w drops to 0 and leaves the basis; our next basic feasible solution is

$$\begin{bmatrix} y_2 \\ y_5 \\ y_4 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10/26 \\ 13/26 \\ 17/26 \\ 1/26 \end{bmatrix}.$$

This is what we were asked to find.