Chvátal's t_0-tough conjecture

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Let $k(G)$ denote the number of components in a graph G.

Theorem 1

If G is a hamiltonian graph, then

$$k(G - S) \leq |S|$$

for every nonempty proper subset S of $V(G)$.

So, if G is hamiltonian,

$$\frac{|S|}{k(G - S)} \geq 1$$

for every nonempty proper subset S of $V(G)$.

(Chvátal 1973) A noncomplete graph G is t-**tough** if

$$\frac{|S|}{k(G-S)} \geq t$$

for every vertex cut S of G. The **toughness** $t(G)$ of G is the maximum t for which G is t-tough. (We take $t(K_n) = \frac{n-1}{2}$.)

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Let $\alpha(G)$ denote the independence number of G.

Theorem 2 (Chvátal 1973/Matthews & Sumner 1984)

For every noncomplete graph G,

$$\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{\kappa(G)}{2}.$$

If G is clawfree, then $t(G) = \frac{\kappa(G)}{2}$.

As we've seen, every hamiltonian graph G is 1-tough since $\frac{|S|}{k(G-S)} \geq 1$ for every nonempty proper subset S of $V(G)$. However, as noted by Chvátal, the converse is not true.

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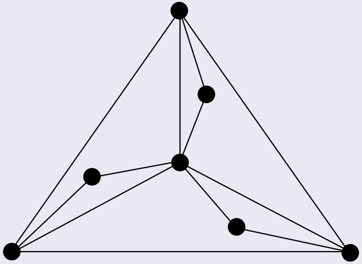
Example.

H:



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H is not hamiltonian

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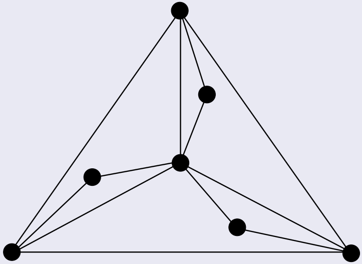

H is 1-tough

H is not hamiltonian

Example.

The Petersen graph P is 1-tough. In fact, $t(P) = \frac{4}{3}$.

Note: We know that high connectivity does not imply hamiltonicity, e.g., $K_{m,n}$ with $m < n$.

Conjecture 3 (Chvátal 1973)

There exists t_0 such that every t_0-tough graph is hamiltonian.

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Conjecture 3 (Chvátal 1973)

There exists t_0 such that every t_0-tough graph is hamiltonian.

Theorem 4 (Chvátal 1973)

There exist infinitely many $\frac{3}{2}$-tough nonhamiltonian graphs.

Conjecture 5 (Chvátal 1973)

Every t-tough graph with $t > \frac{3}{2}$ is hamiltonian.

Theorem 6 (Thomassen 1978)

There exist infinitely many nonhamiltonian graphs G with $t(G) > \frac{3}{2}$.

A k-**factor** of a graph G is a spanning k-regular subgraph of G.

Note: Every hamiltonian graph has a 2-factor but not conversely, e.g., the Petersen graph.

Theorem 7 was conjectured by Chvátal in 1973.

Theorem 7 (Enomoto, Jackson, Katerinis, Saito 1985)

Let G be a k-tough graph of order n with $n \geq k+1$ and kn even. Then G has a k-factor.

Theorem 8 (Enomoto, Jackson, Katerinis, Saito 1985)

Let $k \geq 1$. For any $\epsilon > 0$ there exists a $(k-\epsilon)$-tough graph of order n with $n \geq k+1$ and kn even with no k-factor.

Corollary 9 (Enomoto, Jackson, Katerinis, Saito 1985)

Every 2-tough graph has a 2-factor. Furthermore, for any $\epsilon > 0$ there exist infinitely many $(2-\epsilon)$-tough graphs with no 2-factor.

2-tough Conjecture

Every 2-tough graph is hamiltonian, i.e., $t_0 = 2$.

The truth of the 2-tough conjecture would imply:

Theorem 10 (Fleischner 1974)

The square of every 2-connected graph is hamiltonian.

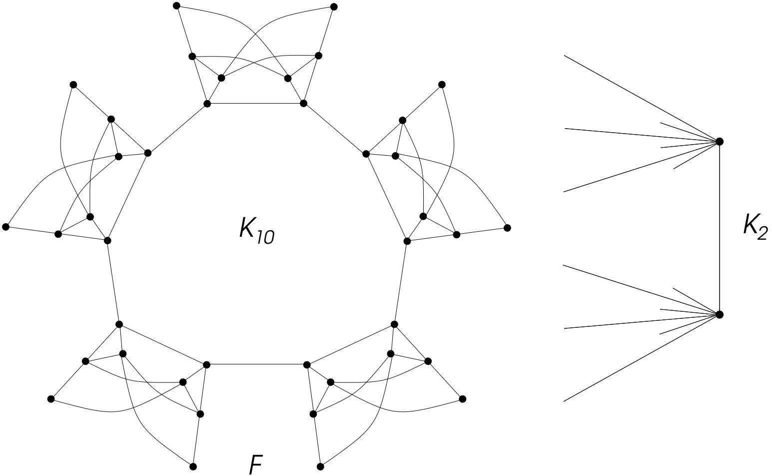
Conjecture 11 (Matthews & Sumner 1984)

Every 4-connected clawfree graph is hamiltonian.

Conjecture 12 (Thomassen 1986)

Every 4-connected line graph is hamiltonian.

Alas, the Bauer-Broersma-Veldman graph $G = F \vee K_2$:



G is 2-tough but not hamiltonian.

Theorem 13 (Bauer, Broersma, Veldman 2000)

For every $\epsilon > 0$ there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable (and, consequently, nonhamiltonian) graph.

So, the "2-tough conjecture" is not true. But Chvátal's original conjecture remains open.

Conjecture 14 (Chvátal 1973)

There exists t_0 such that every t_0-tough graph is hamiltonian.

What is known about Conjecture 14?

- If true, $t_0 \geq \frac{9}{4}$
- True for a number of well-studied graph classes

Example.

Theorem 15

Let G be a planar graph with $t(G) > \frac{3}{2}$. Then G is hamiltonian.

Note: This follows from a well-known result of Tutte that 4-connected planar graphs are hamiltonian and the fact that $\kappa(G) \geq 2t(G)$.

Note: $\frac{3}{2}$-toughness does not imply hamiltonicity.

Some other known results:

Theorem 16 (Keil 1985)

Every 1-tough interval graph is hamiltonian.

Theorem 17 (Balakrishnan & Paulraja 1986)

Every 1-tough clawfree chordal graph is hamiltonian.

Theorem 18 (Kratsch, Lehel & Müller 1996)

Every $\frac{3}{2}$-tough split graph is hamiltonian.

Note: There is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $t(G_n) \to \frac{3}{2}$.

Theorem 19 (Deogen, Kratsch & Steiner 1997)

Every 1-tough cocomparability graph is hamiltonian.

Theorem 20 (Chen, Jacobson, Kézdy & Lehel 1998)

Every 18-tough chordal graph is hamiltonian.

Note: For every $\epsilon > 0$ there are chordal graphs with toughness $\frac{7}{4} - \epsilon$ that are not hamiltonian.

Theorem 21 (Böhme, Harant & Tkác 1999)

Let G be a chordal, planar graph with $t(G) > 1$. Then G is hamiltonian.

Note: 1-toughness does not ensure hamiltonicity.

Theorem 22 (Broersma, Xiong & Yoshimoto 2007)

Every $\frac{k+1}{3}$-tough k-tree is hamiltonian for $k \geq 2$.

Note: For $k \geq 3$ there are infinite families of 1-tough k-trees that are not hamiltonian.

Theorem 23 (Kaiser, Král & Stacho 2007)

Every $\frac{3}{2}$-tough spider (intersection) graph is hamiltonian.

Note: This includes the result of Kratsch, Lehel, & Müller.

Theorem 24

Every $\frac{5}{2}$-tough clawfree graph of minimum degree at least 6 is hamiltonian.

Note: This follows from Theorem 25

Theorem 25 (Kaiser & Vrána, Preprint)

Every 5-connected clawfree graph of minimum degree at least 6 is hamiltonian-connected.

Corollary 26

Every 6-connected clawfree graph is hamiltonian, i.e., every 3-tough clawfree graph is hamiltonian.

Some open questions:

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1. Are 2-tough clawfree graphs hamiltonian? This is equivalent to the conjecture of Matthews and Sumner (1984) that 4-connected clawfree graphs are hamiltonian.
2. Are $\frac{7}{4}$-tough chordal graphs hamiltonian? Or, more likely, are 2-tough chordal graphs hamiltonian?
3. Are $\frac{3}{2}$-tough maximal planar graphs hamiltonian?

   For every $\epsilon > 0$ there are maximal planar $(\frac{3}{2} - \epsilon)$-tough nonhamiltonian graphs.

Some open questions (continued):

4. What about triangle-free graphs? Are 2-tough triangle-free graphs hamiltonian?

   It is conjectured (Bauer, van den Heuvel, Schmeichel 1996) that for all $\epsilon > 0$, there exists a $(2 - \epsilon)$-tough triangle-free graph with no 2-factor.

   There is an infinte class (Ferlund 1993) of nonhamiltonian triangle-free graphs whose toughness is at least $\frac{5}{4}$.

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5. What about special families of chordal graphs? For example, *k*-trees, strongly chordal graphs, caterpillar intersection graphs.
6. What about Wei's branching number?