EXAMPLES ARE FOREVER

SIEMION FAJTLOWICZ, UNIVERSITY OF HOUSTON

She had a million dollars worth of nickels and dimes
She sat around and counted them all a million times
(Cab Calloway, "Minnie the Moocher")

I have known Paul Erdös for about a quarter of a century, but until our last meeting in January of '96 I hardly remember an occasion which would not reveal something new about him. It is not because he was a more complex (he was just more gifted) person than many others, but because he was more open. Unlike one of Larry Niven's heros who learned at a very young age to raise an emotional shield to protect his mind from intruders, the otherwise precocious Paul Erdös has never acquired this ability. For many people this created a problem they did not know how to cope with. For others it was a refreshing experience.

Erdös was born to travel. We had once a dinner with a group of his collaborators from Texas A&M, and one of them said that his relative was a travelling salesman in Alaska. This instantly evoked Erdös's enthusiastic response: "what a great way to make a living", People seem to think that he was travelling to do mathematics, but I doubt it.

I talked to him for the first time shortly after the death of his mother, who until then was a frequent companion on his trips. He must have loved her very much, but he needed her company for other reasons too. Erdös lived by simple wisdoms of old books, and one of them insists that " *you should never eat alone nor should you ever embark by yourself into a long journey. In a long trip, the most insignificant companion can be helpful.*" Particularly, I would add, if you can't or refuse to learn to tie your shoelaces. His life must have been miserable after the death of his mother and he began to take at that time enormous amounts of anti-depressant pills. I often visualized him as Sinbad the Sailor as portrayed by Paul Klee: a slim man desperately facing inclements of the weather and sea monsters. When we waited for his arrival, Francoise Ulam said: " *perhaps now, Paul will become more independent.*" Paul was about sixty at the time, but he did.

I owed some mathematical debts to Mycielski, but perhaps the greatest one was that he was encouraging me (during the same first lunch with Erdös) to ask him whether a function of two variables which is a polynomial for each fixed value of variables, must itself be a polynomial. I was asking this question of everyone I could, for at least two or three years (in Boulder alone, I am sure, I discussed it with both Ehrenfeucht and Ulam) but it took Erdös not more than one lunch to figure out that for the rationals the answer is "no", and he invented a striking counterexample, which led to our first joint paper. We later solved this problem for arbitrary fields, but even the positive part of the solution was inspired by the idea of his simple, yet very clever example.

We later had many lunches like that, and I would not trade some of them for a dinner with Claudia Schiffer. Mrs. Ulam told us that for a while he was travelling with a female companion - a physicist, but this idyll did not last very long. When Mrs. Ulam asked why, he explained that on the beginning everything was going smoothly, but then: "*you know, she started this nonsense about touching.*" Well, *all good things must end sometimes, except* - as Erdös used to say - *mathematics.* Erdös was one of the few people who did not have any interest in sex, and I thought that it was admirable, how open and complex-free he was about it; he was very comfortable with himself in general. But he authentically liked and understood women. He was one of the few mathematicians who after divorces, did not break off friendships with spouses of his collaborators, and he was always concerned about widows of his deceased friends.

He was habitual complainer, but there were very few things he was truly bitter about, save the subject of the war. I did not have impression that he was that much concerned about not getting enough credit for the elementary proof of the Prime Number Theorem, and this one must have been for him like the diamond car in the collection of Minnie the Moocher. Fortunately, because in his childish lingo, "rich" meant plenty of results. But he did not have much forgiveness for those who in his words "*wanted to starve me to death, because I refused to a take a permanent teaching post.*" He was equally, if not more, irritated if some of his favorite collaborators were treated unfairly. When his friend was refused tenure by a very well known school, Erdös wrote that the university just earned itself a footnote in the history of mathematics.

When I became interested in the writing of conjecture-making programs our conversations acquired a new dimension. I kept asking mathematicians, what makes a good conjecture and often they would looked startled. Many others suggested that I should ask Erdös. I had done so already, and Erdös at least did not look surprised. However his answer "let's leave it to Radamantys," was not much help in writing of the program. On the other hand, there was a very good point to his answer. Many mathematicians assume that good conjectures must be difficult, but eventually the significance of a problem seems to be fairly independent of its solution. Euclid's algorithm and Gaussian Elimination method are perhaps the best examples. Within the last thirty years these two simple old ideas had great impact on algebraic geometry. Hilbert's Decision Problem proved to be much easier then his Tenth Problem yet I think, it had more influence on mathematics. Erdös was sometimes criticized for working on too many problems instead of concentrating on a few big ones. We'd better leave it to Radamantys.

Later I made up my mind that one should always make the strongest conjecture to which one does not know a counter-example, and I asked for his opinion. He thought that conjectures should be moderate. Paul Erdös, was of course, offering money, sometimes quite boldly, for his conjectures,

It is then, I started to believe, that for psychological reasons, computers eventually will be better in making of conjectures than humans. Well, at least most of them, because one of the very few people who had answered my question was John Conway, who without any hesitation said that a good conjecture should be outrageous.

Nevertheless I have learned quite a bit about Erdös asking him this question. His favorite mathematician was Cantor. Euler, and even Gauss who I would expect to be his first choices were far beyond. He thought that the ability to ask a good question is independent from the problem-solving skills. He suggested that one should work on conjectures for their own sake and quoted Gauss's studies, preceding his prime number conjecture, as an example.

Once when he called during his usual winter tours of the southern states, I asked him whether it was a known conjecture that for every prime p there is a p-term arithmetic progression of primes starting with p. The connection was very bad, Erdös kept saying something about Hardy and Littlewood and I assumed that they thought about it. Actually, he was telling me that this does not follow from their very general conjecture, and I knew that he must have liked the idea, because he changed his travel plans and offered on the spot to come to Houston and stay for a week.

His comments on this problem, some with collaboration of my two students Ermelinda DeLaVina and Craig Larson, are described in the "Written on the Wall" - a list of conjectures of Graffiti. One of the most appealing aspects of being around Erdös was that one could almost feel the presence of mathematicians born in 19th century.

That was also the first occasion when I've seen him extensively using a hand calculator and I remember him saying (on another occasion) that it would be really something if one could generate a lot of examples of truly large primes. Computers would have gained a lot in his eyes, if they could accomplish this.

Number-theoretical conjectures of the program were some of his most favorite, and often during his visits he would start with computing by hand the residue of the graph G of consecutive integers $2..n$ in which two vertices are adjacent if and only if they are not relatively prime. The original conjecture was proved by Favaron, Maheo and Sacle. "Written on the Wall" contains some of his joint results with Staton concerning this problem and our joint follow-up conjectures. The first of these was his question whether there is a constant c ≥ 0 such that the residue of $G \leq (1-c)\ \frac{n}{logn}$. It is interesting that in a sense we do not have independence theory, in spite of very advanced Ramsey and matching theories. The third crucial example is the problem of distribution of primes, because $\pi(n)$ is the independence number of G. The results of Erdös and Staton use the Prime Number Theorem. Perhaps at the very least one could find a graph-theoretical generalization of Euclid's proof of infinitude of primes.

Once I referred to the conjecture 448 of Graffiti, which led to these considerations ,by its number, and he looked startled. I said that he certainly must have made more conjectures than the program, but he denied this. In 1996 he thought that it would be a real breakthrough if computer could defeat Kasparov. I asked whether it would matter if this was accomplished just by building faster machines, and his answer was: "no."

In the early nineties, during his visit in Houston, he crunched one by one probabilistic counter-examples to several conjectures of Graffiti involving lower bounds for the independence number in terms of the number of horizontal edges and points at even distance from a given vertex, and then he commented: " *the machine is obsessed with this idea of even and horizontal.*" Soon after this, I stated in my Kalamazoo talk simple bounds for the independence and the chromatic number in terms of exactly these concepts, and I noticed a (friendly) smile of disbelief on his face. The bounds were similar to conjectures which he had refuted. Usually, however he had a different attitude. When he disproved a conjecture, he would work on modification of premises to get a positive result. This is exactly what he did in his paper with Pach and Spencer, publishing the first result on a conjecture of a computer program, before I even started to maintain "Written on the Wall." Some time before that we discussed a counter-example to a very early conjecture of Graffiti, which he thought was totally ridiculous, but he modified it and suggested that we write a joint paper with the program.

Our work with DeLaVina on the case of equality for a similar conjecture, led us to a definition of Ramseyan properties, i.e., classes of graphs in which independence grows to infinity together with the number of vertices. During his last visit in Houston, Erdös became interested in this problem for triangle-free graphs and established the first lower bounds. We have solved the problem asymptotically and in the process we formed a conjecture that every triangle-free graph which has more than $qR(p,3)$ vertices, where $R(k,l)$ is the corresponding Ramsey number, contains a $(p+1)$ -vertex independent set, or a $(q+1)$-element set in which every two vertices are at distance at least 3. This conjecture is still open, but before we announced it, I suggested that according to the principle of the strongest conjecture we should make the appropriate conjecture for K_p-free graphs. This time Erdös agreed, even though we did not examine a single non-trivial example corresponding to the case $p > 3$. The more general conjecture was soon disproved by Bollobas and Riordan, but I don't think Erdös had any regrets. The solution was obtained by a very complicated probabilistic argument.

Working with Erdös and Graffiti, it was interesting to pay attention to differences between the human and the program-invented mathematics. Erdös pointed out to me that Graffiti has never made as complex conjectures as some of his own. That was in early stages of the program, and I wonder whether this is still true. It seems that it is very difficult to show that computers in principle can not do something that a human mathematician can. Whatever the differences, Erdös and Graffiti had one point in common: their ideas were deeply rooted in examples.

I once wrote about Graffiti on the invitation of an editor of "Computers and Philosophy." The story was never published, because I did not see point of making the changes requested by the editor, but I told Erdös that to illustrate the idea of the program I described there a planet where mathematicians accept as true all open conjectures. Mathematicians on this planet do not prove theorems but work on refutations of the most challenging conjectures.

It turned out that Erdös has invented a similar story. He told me about a planet with a monument built for the first mathematician who suggested that there is a need for proving the Riemann Hypothesis. I think that on such planets, mathematics can be as reliable as on ours, at least for engineering purposes, but I did not have in mind what later became to be called experimental mathematics. The main task of these extraterrestrial mathematicians would be to decide which conjectures deserve the most attention. Actually, perhaps even on Earth, the main appeal of theorem-proving is not the reliability of results. I think that one of the main functions of theorem proving is that it forces us to invent concepts to make conjectures to see familiar objects and examples in a new light.

Babai once pointed out to me that there are no mathematical objects bearing Erdös's name. I do not know about Erdös's planet, but on mine there is a monument for the mathematician who discovered a universal method of refuting conjectures of all kind - the probabilistic method. And he did much more than that. He worked on the details of the method and was nagging his collaborators to write papers about it. He did not stop even when the idea have already began to inspire books, conferences and journals. He invented examples so elusive that they could not even be given his name, and yet they are as concrete, as monuments.

Often, the probabilistic counter-examples, usually of Alon, Erdös and Shearer, were too large to be represented in Graffiti. This prompted me to formulate for myself a couple of problems. One is the conceptual representation of objects. If one could solve this problem in practice one could drastically cut down on the percentage of false conjectures in Graffiti and I do not think one can compensate for this problem by just using larger computers or by packing into the program more examples. Another problem is to propose a computable definition of deterministic objects, i.e., those which exclude random examples and as little else as possible.

A surprising, at least at first, property of the probabilistic method is that if one can show existence of an object using this method then usually almost every object under consideration will have the required property. At least this is so in formal mathematics, because in the one which is done daily, this image is highly misleading. Some of the databases of Graffiti contain a fairly representative (relative to our knowledge of examples) sample of objects, yet very few of them are random. Random objects corresponding to the same probability are very much alike, so including more examples would serve no useful purpose.

Sometimes there is need for examples which are not random. One usually asks in such situation for explicit objects, of say graphs, but it is difficult even to formulate conjectures about them, because from the formalistic point of view almost all graphs are random.

The problem of conceptual representation of objects can be probably solved, at least for the sake of Graffiti-like programs, by conjecturing the value of invariants of conceptual objects, similarly as, I believe, we do it mentally. I do not have any idea how the second problem could be approached, or even how to make it more precise. My first thought was that graphs defined by simple algebraic formulas might do, but then Paley graphs are quasi-random.

During one of Erdös's first visits in Houston, I gave him for bed-side reading Van Vogt's "The Monster," published also under the title "Resurrection." He loved the story and he was bringing it up in conversations years later. It just occurred to me that it might even inspire his well-known $R(6,6)$ story, but that does not matter that much. The point is that he fell in love with an unassuming superman of the future, squaring with alien invaders and using his superior intellect to save Earth. To think about it, Jeff Goldblum in "Independence Day" might look like Erdös if he put some work into the winning smile displayed by Erdös in the "Art of Counting." This smile portrayed his true nature, in spite of the fact that he was often in physical or emotional pain. Sometimes this had impact on his disposition. I mentioned earlier lunches with him, but one or two, I would gladly trade for the company of Minnie the Moocher, and consider it a bargain. Yet, these moments were hardly ever a problem, because for such an unconventional person, Erdös had an unusual gift of settling differences of opinion quickly and gracefully. Anyway, it was difficult to hold against him anything at all. As Minnie the Moocher, Paul Erdös was the roughest, toughest frail, but then, he also had a heart, as big as a whale.