Winter 2012

COMP 6621 Discrete Mathematics of Paul Erdős
Homework Assignment 1
Solutions

Solve the following problems by combinatorial reasoning, arguing that both sides of an identity count the same objects. Solutions that plug in the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and then grind away will get no credit.

1. Express
$$\binom{n}{k}\binom{n-k}{m-k}$$
as another product of two binomial coefficients.

An answer:
$$\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}\binom{m}{k}.$$

A justification: Both sides count the number of pairs (S,T) such that $S \subseteq T \subseteq \{1,2,\ldots,n\}$ and $|T| = m$, $|S| = k$. In the left-hand side count, we choose S first and $T - S$ second; in the right-hand side count, we choose T first and S second.

2. Simplify
$$\sum_{k=i}^{n}\binom{k}{i}\binom{n-k}{j}.$$

An answer:
$$\sum_{k=i}^{n}\binom{k}{i}\binom{n-k}{j} = \binom{n+1}{i+j+1}.$$

A justification: Both sides count the number of integer sequences $a_1, a_2, \ldots a_{i+j+1}$ such that $0 \le a_1 < a_2 < \ldots a_{i+j+1} \le n$. In the left-hand side count, these sequences are grouped by the value of a_{i+1}.