THE SECOND MOMENT METHOD IN COMBINATORIAL ANALYSIS

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The *second moment method* is applied to a variety of combinatorial problems some of which are described here.

1. The set $\{1,2,\ldots,2n\}$ is divided into two classes, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1,\ldots,b_n\}$. Let $M_k(A) = \sum_{a_i - b_j = k} 1$, and

   $f(n) = \min_A \max_k M_k(A)$. We find lower bounds for $f(n)$.

2. Let $f(n)$ be the least number of integers $a_1, a_2, \ldots, a_{f(n)}$ such that the integers $1,2,\ldots,n$ can all be written as the sum of four a_i's. We find lower bounds for $f(n)$.

3. Let θ_i be real and $\sum_{i=1}^{n} \theta_i^2 = 1$. We show that, given any integer $k > 1$, there exist integers x_i, $i = 1,2,\ldots,n$, not all zero, such that $|x_i| < k$ and

$$|\theta_1 x_1 + \ldots + \theta_n x_n| \le \sqrt{\frac{k^2 - 1}{k^{2n} - 1}} .$$

4. Given m equations in $n > m$ unknowns:

$$\begin{aligned} a_{11}\, x_1 + \ldots + a_{1n}\, x_n &= 0 \\ &\vdots \\ a_{m1}\, x_1 + \ldots + a_{mn}\, x_n &= 0, \end{aligned}$$

with integer coefficients a_{ij} bounded, in absolute value, by A. Then there exists a non-trivial integral solution $x_1, x_2, \ldots, x_n$ with

$$\max_{i=1,\dots,n} |x_i| < (c\sqrt{n}A)^{\frac{m}{n-m}} .$$

As special cases of the method used above we have:

(i) Given n coins, each weighing 0 or 1. Question. How many subsets msut be weighed in order to determine the weights of all the coins?
Answer. At least $\dfrac{n \log 4}{2 + \log n}$.

(ii) Given k positive integers $a_1 < a_2 < \dots < a_k$, with the $2^k - 1$ non-empty subsets having distinct sums, we show that
$$\sum_{i=1}^{k} a_i^2 \geq \frac{4^k - 1}{3} .$$