RATIONAL BEHAVIOUR
AND COMPUTATIONAL COMPLEXITY

by

V. Chvátal

Technical Report SOCS-78.9
November 1978

School of Computer Science
McGill University
Montreal

Mathematical models of economic behaviour sometimes suffer from lack of realism on at least two counts: they assume that all the participants have access to a complete information about their environment, and that they make use of this information in a perfectly rational way. There is more to the second assumption than meets the eye. Finding an optimal response to an implicit stimulus may be next to impossible even for a well-informed subject: the stimulus itself may be extremely difficult to recognize. We shall illustrate this point on the case of *linear production games* studied by Owen [2].

A linear production game is specified by an $m \times n$ matrix A, an $m \times p$ matrix B and a row vector c of length n. The m rows of A and B correspond to *resources*, the n columns of A and the n components of c correspond to *activities*, and the p columns of B correspond to *players*. The matrix A is the *technology matrix*: each of its entries a_{ij} specifies the amount of resource i required to maintain activity j at a unit level. The matrix B is the *resource matrix*: each of its entries b_{ik} specifies the amount of resource i owned by player k. The vector c is the *revenue vector*: each of its components c_j specifies the net revenue obtained from maintaining activity j at a unit level.

The players pool their resources, maintain the activities at certain levels and divide the resulting net revenue among themselves. If x_j denotes the level of activity j then

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}x_j &\le \sum_{k=1}^{p} b_{ik} && \text{for each resource } i, \\ x_j &\ge 0 && \text{for each activity } j. \end{aligned} \tag{1}$$

If d_k denotes the dividend paid to player k then

$$\sum_{k=1}^{p} d_k = \sum_{j=1}^{n} c_j x_j \ . \tag{2}$$

Conversely, every pair of vectors $x = (x_j)$ and $d = (d_k)$ satisfying (1) and (2) describes a conceivable state of affairs. A set S of players has a good reason not to participate in this program if this coalition can maintain the activities at levels x_j^* $(1 \le j \le n)$ and split the resulting net revenue into dividends d_k^* $(k \in S)$ such that $d_k^* > d_k$ for every player k in S. Clearly, this is the case if and only if there are numbers $x_1^*, x_2^*, \ldots, x_n^*$ such that

$$\sum_{j=1}^{n} a_{ij} x_j^* \le \sum_{k \in S} b_{ik} \qquad \text{for each resource } i \ ,$$

$$x_j^* \ge 0 \qquad \text{for each activity } j \ ,$$

$$\sum_{j=1}^{n} c_j x_j^* > \sum_{k \in S} d_k \ .$$

This condition can be stated more succintly as

$$\sum_{k \in S} d_k < v(S) \tag{3}$$

with $v(S)$ standing for the optimal value of the linear programming problem

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le \sum_{k \in S} b_{ik} \qquad (i = 1,2,\ldots,m)$$

$$x_j \ge 0 \qquad (j = 1,2,\ldots,n) \ .$$

We shall say that a state represented by x and d is *unstable* if (3) holds for at least one set S.

It may seem that an unstable state cannot persist: after all, the members of the coalition S can bargain with the remaining players for higher dividends or, if need be, split off altogether. And yet this argument is faulty. The members of S may be *unaware* of their bargaining power even if they have access to all the relevant data: detecting instability in a game with a few hundred players and a few hundred resources may be virtually impossible. More precisely, we shall prove that detecting instability in linear production games is no easier than finding zero-one solutions to systems of linear inequalities. The latter task is notoriously difficult. In fact, a theorem proved a few years ago by Cook [1] supports the popular belief that there exists no efficient algorithm for solving such problems.

We shall consider those linear production games which involve only one activity. In addition, we shall assume that maintaining this activity at a unit level requires a unit amount of each resource and brings in a unit revenue. Under these assumptions, we have

$$v(S) = \min_i \sum_{k \in S} b_{ik}$$

for every set S. In a stable state, the activity must be maintained at its optimum level,

$$x = \min_i \sum_{k=1}^{p} b_{ik} .$$

We shall assume that the total net revenue is split evenly among the p players, so that

$$d_k = x/p$$

for every k. Even in this rather special situation, recognizing instability is no easier than finding zero-one solutions $x_1, x_2, \ldots, x_N$ to systems of linear inequalities

$$\sum_{j=1}^{N} a_{ij}^* x_j > b_i^* \quad (i = 1,2,\ldots,M) \ . \tag{4}$$

To prove this claim, we first consider the $(M+1) \times (N+2)$-matrix $W = (w_{ik})$ defined by

$$w_{ik} = \begin{cases} a_{ik}^* & \text{if } 1 \le i \le M \text{ and } 1 \le k \le N \\ -b_i^* & \text{if } 1 \le i \le M \text{ and } k = N+1 \\ b_i^* - \sum_{k=1}^{N} a_{ik}^* & \text{if } 1 \le i \le M \text{ and } k = N+2 \\ 0 & \text{if } i = M+1 \text{ and } 1 \le k \le N \\ 1 & \text{if } i = M+1 \text{ and } k = N+1 \\ -1 & \text{if } i = M+1 \text{ and } k = N+2 \end{cases}$$

It is not difficult to verify that (4) has a zero-one solution if and only if

$$\min_i \sum_{k \varepsilon S} w_{ij} > 0 \tag{5}$$

for some set S. Next, we choose a positive number t large enough to make every

$$b_{ik} = w_{ik} + t$$

nonnegative. Consider a linear production game in which player k owns b_{ik} units of resource i and receives a dividend of $d_k = t$. This state is unstable if and only if

$$\sum_{k\varepsilon S} d_k < \min_i \sum_{k\varepsilon S} b_{ik} \tag{6}$$

for some coalition S. Since (6) is clearly equivalent to (5), the state is unstable if and only if (4) has a zero-one solution.

Our observations suggest that the concept of "rational behaviour" has to be formalized carefully. It would be unreasonable to assume that a rational player can always recognize the presence of a coalition S satisfying (3). Instead, one could perhaps assume that each player is endowed with a *heuristic* which may or may not find such coalitions. When each player's heuristic fails, the state has to be declared stable even if (3) does hold for some S. It is tempting to speculate about similar phenomena in other behavioural sciences.

REFERENCES

[1] S.A. COOK, "The complexity of theorem-proving procedures", Conference Record of Third ACM Symposium on Theory of Computing (1970), 151-158.

[2] G. OWEN, "On the core of linear production games", Mathematical Programming 9(1975), 358-370.