ANTIMATROIDS, BETWEENNESS, CONVEXITY

Vašek Chvátal[1]
Canada Research Chair in Combinatorial Optimization,
Department of Computer Science and Software Engineering,
Concordia University, Montréal, Québec H3G 1M8, Canada

1 Introduction

Korte and Lovász [12, 13] founded the theory of *greedoids*. These combinatorial structures characterize a class of optimization problems that can be solved by greedy algorithms. In particular, greedoids generalize *matroids*, introduced earlier by Whitney [16]. *Antimatroids*, introduced by Dilworth [3] as particular examples of semimodular lattices, make up another class of greedoids.

Antimatroids are related to abstract convexity; let us explain how. Kay and Womble [11] defined a *convexity space* on a ground set E as a tuple $(E, \mathcal{N})$, where $\mathcal{N}$ is a collection of subsets of E such that $\emptyset \in \mathcal{N}$, $E \in \mathcal{N}$, and $\mathcal{N}$ is closed under intersections. Members of $\mathcal{N}$ are called *convex sets*. The *convex hull* of a subset X of E is defined as the intersection of all convex supersets of X and is denoted by $\tau_{\mathcal{N}}(X)$. Independently of each other, Edelman [6] and Jamison [9] initiated the study of convexity spaces $(E, \mathcal{N})$ with the *anti-exchange property*

> if $X \subseteq E$ and y, z are distinct points outside $\tau_{\mathcal{N}}(X)$,
> then at most one of $y \in \tau_{\mathcal{N}}(X \cup \{z\})$ and $z \in \tau_{\mathcal{N}}(X \cup \{y\})$ holds true.

Jointly [7], they proposed to call such convexity spaces *convex geometries.* An antimatroid is a tuple $(E, \mathcal{F})$ such that $(E, \{E - X : X \in \mathcal{F}\})$ is a convex geometry.

In the present paper, we deal exclusively with finite ground sets. Our starting point are two examples of antimatroids. One of these arises from double shelling of a poset (example 2.4 in Chapter III of the monograph [14]) and the other from simplicial shelling of a triangulated graph (example 2.7 in Chapter III of [14]). Let us describe them in terms of convex geometries.

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In the first example, given a partially ordered set $(E, \preceq)$, we let $\mathcal{N}$ consist of all subsets K of E such that

$$a, c \in K,\ a \prec b \prec c \ \Rightarrow\ b \in K;$$

the resulting tuple $(E, \mathcal{N})$ is a convex geometry. In the second example, given an undirected graph with a vertex-set E, we let $\mathcal{N}$ consist of all subsets K of E such that

$$a, c \in K,\ b \text{ is an interior vertex of a chordless path from } a \text{ to } c \ \Rightarrow\ b \in K.$$

The resulting tuple $(E, \mathcal{N})$ is a convexity space, but not necessarily a convex geometry: for instance, take a chordless cycle through four vertices as the graph and consider the convex set X consisting of two adjacent vertices in this graph. Nevertheless, if the graph is *triangulated*, meaning that it contains no chordless cycle through four or more vertices, then $(E, \mathcal{N})$ is a convex geometry. To elucidate this point, we appeal to a characterization of convex geometries that involves the notion of an *extreme point* of a convex set K, defined as a point b of K such that $K - \{b\}$ is convex.

Nine equivalent characterizations of convex geometries are stated in [9]; equivalence of the following four is proved in [14], Chapter III, Theorem 1.1.

Fact 1. *For every convexity space $(E, \mathcal{N})$, the following four propositions are logically equivalent:*

(G1) *$(E, \mathcal{N})$ has the anti-exchange property.*

(G2) *If $X \in \mathcal{N}$ and $X \neq E$, then $X \cup \{y\} \in \mathcal{N}$ for some y in $E - X$.*

(G3) *Every set in $\mathcal{N}$ is the convex hull of its extreme points.*

(G4) *Every subset X of E contains a unique minimal subset Y such that $\tau_{\mathcal{N}}(Y) = \tau_{\mathcal{N}}(X)$.*

□

To see that our second example has property (G3), we invoke a theorem of Dirac [4]. There, a vertex is called *simplicial* if its neighbours are pairwise adjacent.

Fact 2. *For every finite undirected graph G, the following three propositions are logically equivalent:*

- *G is triangulated.*
- *Every minimal cutset in G is a clique.*
- *Every induced subgraph of G either includes two nonadjacent simplicial vertices or is complete.*

□

A corollary of this theorem (stated and proved in [8] as Theorem 3.2) asserts that

> *every non-simplicial vertex in a triangulated graph*
> *lies on a chordless path between two simplicial vertices;*

since the extreme points of E in the second example are precisely the simplicial vertices of G, it follows that every point of E lies in the convex hull of at most two extreme points of E.

Each of these two examples of convex geometries is constructed through the intermediary of betweenness in the underlying structure. In the first example, we may say that b lies between a and c if, and only if, $a \prec b \prec c$ or $c \prec b \prec a$; in the second example, we may say that b lies between a and c if, and only if, b is an interior vertex of a chordless path from a to c; in either case, a subset K of E is convex if and only if

$$a, c \in K,\ b \text{ lies between } a \text{ and } c \ \Rightarrow\ b \in K.$$

This construction generalizes: as in [1], every ternary relation $\mathcal{B}$ on a finite ground set E defines a convexity space $(E, \mathcal{N}_{\mathcal{B}})$ by

$$\mathcal{N}_{\mathcal{B}} = \{K \subseteq E : a, c \in K,\ (a, b, c) \in \mathcal{B} \ \Rightarrow\ b \in K\}.$$

Our objective is to characterize a nested pair of classes of ternary relations $\mathcal{B}$ on finite ground sets E such that the corresponding classes of convexity spaces $(E, \mathcal{N}_{\mathcal{B}})$ consist exclusively of convex geometries, include all convex geometries that arise from double shelling of a poset, and include all convex geometries that arise from simplicial shelling of a triangulated graph.

2 Results

Note that, for every ternary relation $\mathcal{B}$ on a finite ground set, every set K in $\mathcal{N}_\mathcal{B}$, and every point b of K,

$$K - \{b\} \in \mathcal{N}_\mathcal{B} \quad \Leftrightarrow \quad \text{there are no points } a, c \text{ of } K \text{ such that } (a,b,c) \in \mathcal{B}.$$

This observation allows us to extend the definition of extreme points: given an arbitrary, not necessarily convex, subset X of the ground set and given an arbitrary point b of X, we shall say that b is an *extreme point of* X if, and only if, there are no points a, c of X such that $(a,b,c) \in \mathcal{B}$. The set of all extreme points of X will be denoted by $\mathrm{ex}_\mathcal{B}(X)$.

In addition, note that $\mathcal{N}_\mathcal{B}$ does not change if $\mathcal{B}$ is made symmetric by including (c,b,a) in $\mathcal{B}$ whenever (a,b,c) is in $\mathcal{B}$; it does not change either if all triples (b,b,b), (b,b,c), (a,b,b) are removed from $\mathcal{B}$. Note also that $\mathcal{N}_\mathcal{B}$ includes all singletons $\{a\}$ with $a \in E$ (Kay and Womble [11] designate such convexity spaces T_1) if and only if $\mathcal{B}$ includes no triple (a,b,a) with $b \neq a$. We will restrict our attention to ternary relations $\mathcal{B}$ such that

$$(a,b,c) \in \mathcal{B} \;\Rightarrow\; (c,b,a) \in \mathcal{B} \text{ and } a,b,c \text{ are pairwise distinct;}$$

any such $\mathcal{B}$ will be called a *strict betweenness*.

Theorem 1. *For every strict betweenness $\mathcal{B}$ on a finite ground set E, the following two propositions are logically equivalent:*

(i) *For all subsets X of E and all x_1, x_2, x_3 in X such that $(x_1,x_2,x_3) \in \mathcal{B}$, there are $\overline{x}_1, \overline{x}_3$ in $\mathrm{ex}_\mathcal{B}(X)$ such that $(\overline{x}_1, x_2, \overline{x}_3) \in \mathcal{B}$.*

(ii) $(a,b,c_2), (c_1,c_2,c_3) \in \mathcal{B} \Rightarrow (a,b,c_1) \in \mathcal{B}$ *or* $(a,b,c_3) \in \mathcal{B}$ *or* $(c_1,b,c_3) \in \mathcal{B}$.

□

Following [11], a convexity space is said to have *Carathéodory number d* if, and only if, d is the smallest positive integer with the following property:

if a point lies in the convex hull of a set X,
then it lies in the convex hull of a subset X' of X such that $|X'| \leq d$.

Theorem 1 characterizes a class of ternary relations $\mathcal{B}$ on finite ground sets E such that the corresponding class of convexity spaces $(E, \mathcal{N}_\mathcal{B})$ consists exclusively of convex geometries with Carathéodory number at most 2. However, it does not characterize all such relations: for instance, if $E = \{a, b, c_2, c_3\}$ and

$$\mathcal{B} = \{(a, b, c_2), (c_2, b, a), (a, c_2, c_3), ((c_3, c_2, a)\},$$

then $(E, \mathcal{N}_\mathcal{B})$ is a convex geometry with Carathéodory number 2 and yet $\mathcal{B}$ does not satisfy the conditions of Theorem 1.

Strict order betweenness[15] in a partially ordered set $(E, \preceq)$ is defined by

$$\mathcal{B} = \{(a, b, c) \in E^3\colon\ a \prec b \prec c \text{ or } c \prec b \prec a\};$$

monophonic [10], or *minimal path* [5], betweenness in an undirected graph with a vertex-set E is defined by

$$\mathcal{B} = \{(a, b, c) \in E^3\colon\ b \text{ is an interior vertex of a chordless path from } a \text{ to } c\}.$$

Strict order betweenness satisfies condition (ii) of Theorem 1: it is a straightforward exercise to verify that it satisfies the stronger condition

$$(a, b, c_2), (c_1, c_2, c_3) \in \mathcal{B} \Rightarrow (a, b, c_1) \in \mathcal{B} \text{ or } (a, b, c_3) \in \mathcal{B}.$$

We shall prove that monophonic betweenness, too, satisfies this stronger condition.

Theorem 2. *Let G be a finite triangulated graph and let a, b, c_1, c_2, c_3 be vertices of G. If*

b is an interior vertex of a chordless path between a and c_2 and
c_2 is an interior vertex of a chordless path between c_1 and c_3,

then

b is an interior vertex of a chordless path between a and c_1, or else
b is an interior vertex of a chordless path between a and c_3.

□

Theorem 3. *For every strict betweenness $\mathcal{B}$ on a finite ground set E, the following two propositions are logically equivalent:*

(i) *For all subsets X of E and all x_1, x_2, x_3 in X such that $(x_1, x_2, x_3) \in \mathcal{B}$, there is an $\overline{x}_3$ in $\mathrm{ex}_\mathcal{B}(X)$ such that $(x_1, x_2, \overline{x}_3) \in \mathcal{B}$.*

(ii) *$(a, b, c_2), (c_1, c_2, c_3) \in \mathcal{B} \Rightarrow (a, b, c_1) \in \mathcal{B}$ or $(a, b, c_3) \in \mathcal{B}$.*

□

3 Proofs

Our proof of Theorem 1 parallels a proof of the theorem of Dietrich [2] that characterizes antimatroids in terms of circuits (see also Theorem 3.9 in Chapter III of [14]). It begins with a pair of auxiliary results.

Lemma 1. *Let $\mathcal{B}$ be a strict betweenness on a finite ground set E. If*

$$(a,b,c_2),(c_1,c_2,c_3)\in\mathcal{B} \Rightarrow (a,b,c_1)\in\mathcal{B} \text{ or } (a,b,c_3)\in\mathcal{B} \text{ or } (c_1,b,c_3)\in\mathcal{B},$$

then, with $\mathcal{N}=\mathcal{N}_{\mathcal{B}}$,

$$\tau_{\mathcal{N}}(X) = X\cup\{b: \text{ there are } a,c \text{ in } X \text{ with } (a,b,c)\in\mathcal{B}\}$$

for all subsets X of E.

Proof. Write $X'=\{b: \text{ there are } a,c \text{ in } X \text{ with } (a,b,c)\in\mathcal{B}\}$. Since $\tau_{\mathcal{N}}(X)$ is a convex superset of X, it is a superset of $X\cup X'$; our task reduces to proving that $X\cup X'$ is convex. For this purpose, consider an arbitrary b in E such that $(a,b,c)\in\mathcal{B}$ for some a,c in $X\cup X'$: we are going to prove that $b\in X\cup X'$.

Case 1: $a,c\in X$.

In this case, $b\in X'$ by definition of X'.

Case 2: $a\in X$, $c\in X'$.

By definition of X', there are c_1,c_3 in X with $(c_1,c,c_3)\in\mathcal{B}$; the hypothesis of the lemma with $c_2=c$ guarantees that $(c_1,b,c_3)\in\mathcal{B}$ or $(a,b,c_1)\in\mathcal{B}$ or $(a,b,c_3)\in\mathcal{B}$. But then we are back in Case 1.

Case 3: $a,c\in X'$.

As in Case 2, we find c_1,c_3 in X such that $(c_1,b,c_3)\in\mathcal{B}$ or $(a,b,c_1)\in\mathcal{B}$ or $(a,b,c_3)\in\mathcal{B}$. If $(c_1,b,c_3)\in\mathcal{B}$, then we are back in Case 1; if $(a,b,c_1)\in\mathcal{B}$ or $(a,b,c_3)\in\mathcal{B}$, then we are back in Case 2. □

Lemma 2. *Let $\mathcal{B}$ be a strict betweenness on a finite ground set E. If*

$$(a,b,c_2),(c_1,c_2,c_3)\in\mathcal{B} \Rightarrow (a,b,c_1)\in\mathcal{B} \text{ or } (a,b,c_3)\in\mathcal{B} \text{ or } (c_1,b,c_3)\in\mathcal{B},$$

then $(E,\mathcal{N}_{\mathcal{B}})$ is a convex geometry.

Proof. Write $\mathcal{N}=\mathcal{N}_{\mathcal{B}}$. We will show that the convexity space $(E,\mathcal{N})$ has the anti-exchange property. For this purpose, assume the contrary: there are a subset X of E and distinct points y,z outside $\tau_{\mathcal{N}}(X)$ such that $y\in$

$\tau_{\mathcal{N}}(X \cup \{z\})$ and $z \in \tau_{\mathcal{N}}(X \cup \{y\})$. Since $y \in \tau_{\mathcal{N}}(X \cup \{z\}) - \tau_{\mathcal{N}}(X)$ and $y \neq z$, Lemma 1 guarantees that $(x_y, y, z) \in \mathcal{B}$ for some x_y in X; similarly, $(x_z, z, y) \in \mathcal{B}$ for some x_z in X. But then the hypothesis with $a = x_y$, $b = y$, $c_1 = y$, $c_2 = z$, $c_3 = x_z$ implies that $(x_y, y, x_z) \in \mathcal{B}$, and so $y \in \tau_{\mathcal{N}}(X)$, a contradiction. □

Proof of Theorem 1. To see that (i) implies (ii), set $X = \{a, b, c_1, c_2, c_3\}$ and $x_1 = a$, $x_2 = b$, $x_3 = c_2$ in (i). To show that (ii) implies (i), consider an arbitrary subset X of E and arbitrary x_1, x_2, x_3 in X such that $(x_1, x_2, x_3) \in \mathcal{B}$; let K denote the convex hull of X. Since $x_1, x_2, x_3 \in K$, we have $x_2 \notin \mathrm{ex}_{\mathcal{B}}(K)$, but Lemma 2 guarantees that x_2 belongs to the convex hull of $\mathrm{ex}_{\mathcal{B}}(K)$; now Lemma 1 (with $\mathrm{ex}_{\mathcal{B}}(K)$ in place of X) provides $\overline{x}_1, \overline{x}_3$ in $\mathrm{ex}_{\mathcal{B}}(K)$ such that $(\overline{x}_1, x_2, \overline{x}_3) \in \mathcal{B}$. Finally, Lemma 1 shows that $\mathrm{ex}_{\mathcal{B}}(K) \subseteq X$, and so $\overline{x}_1, \overline{x}_3 \in \mathrm{ex}_{\mathcal{B}}(X)$. □

Proof of Theorem 2. Let P_1 denote the chordless path from a to c_2 that passes through b and let P_2 denote the chordless path from c_1 to c_3 that passes through c_2. Proceeding along P_1 from a to c_2, we label the vertices consecutively as $v_1, v_2, \ldots, v_m$, so that

$$a = v_1, \;\; b = v_s \text{ for some } s \text{ such that } 2 \leq s \leq m-1, \;\; c_2 = v_m;$$

proceeding along P_2 from c_1 to c_3, we label the vertices consecutively as w_1, $w_3, \ldots, w_n$, so that

$$c_1 = w_1, \;\; c_2 = w_t \text{ for some } t \text{ such that } 2 \leq t \leq n-1, \;\; c_3 = w_n.$$

We claim that

($\star$) none of $v_1, v_2, \ldots, v_{m-2}$ has a neighbour w_i with $i < t$ or
none of $v_1, v_2, \ldots, v_{m-2}$ has a neighbour w_j with $j > t$.

To justify this claim, assume the contrary: there are edges $v_k w_i$ and $v_\ell w_j$ with $k, \ell \leq m-2$ and $i < t$, $j > t$. Choose them so that $|k - \ell|$ is minimized (we may have $k = \ell$) and, subject to this constraint, i is maximized and j is minimized; let P denote the segment of P_1 that stretches between v_k and v_ℓ. Now v_k is the only vertex on P that has a neighbour in $\{w_1, w_2, \ldots, w_{t-1}\}$ and v_ℓ is the only vertex on P that has a neighbour in $\{w_{t+1}, w_{t+2}, \ldots, w_n\}$; as $w_t = v_m$ and $k, \ell \leq m-2$, no vertex on P is adjacent to w_t or identical with w_t. It follows that the paths P and $w_i w_{i+1} \ldots w_j$ are vertex-disjoint and that their union induces a chordless cycle through at least four vertices; this contradiction completes the proof of $(\star)$.

After flipping P_2 if necessary, $(\star)$ lets us assume that none of v_1, v_2, $\ldots$, v_{s-1} has a neighbour w_i with $i < t$. Since the walk $v_s v_{s+1} \ldots v_m w_{t-1} \ldots w_2 w_1$ connects b to c_1, some subset of its vertices induces a chordless path P from b to c_1; now b is an interior vertex of the chordless path $v_1 v_2 \ldots v_{s-1} P$ between a and c_1. $\square$

Proof of Theorem 3. To see that (i) implies (ii), set $X = \{a, b, c_1, c_2, c_3\}$ and $x_1 = a$, $x_2 = b$, $x_3 = c_2$ in (i). To show that (ii) implies (i), we shall use induction on $|X|$. If $|X| \leq 2$, then the conclusion is vacuously true. For the induction step, consider arbitrary x_1, x_2, x_3 in X such that $(x_1, x_2, x_3) \in \mathcal{B}$. Setting

$$Z = \{z \in X : (x_1, x_2, z) \in \mathcal{B}\},$$

we shall proceed to prove that $Z \cap \mathrm{ex}_{\mathcal{B}}(X) \neq \emptyset$.

First we claim that, with $\mathcal{N}$ a shorthand for $\mathcal{N}_{\mathcal{B}}$ as usual,

- $\tau_{\mathcal{N}}(X - Z) \cap Z = \emptyset$.

To justify this claim, assume the contrary: there is a triple (c_1, c_2, c_3) in $\mathcal{B}$ such that $c_1 \in X - Z$, $c_2 \in Z$, $c_3 \in X - Z$. But then the hypothesis of the theorem is contradicted by $a = x_1$, $b = x_2$.

Next, let us write $z' \prec z''$ if and only if $z', z'' \in Z$ and there exists a y in $X - Z$ such that $(y, z', z'') \in \mathcal{B}$; note that $z' \prec z'' \Rightarrow z' \neq z''$. We claim that

- $\prec$ is antisymmetric.

To justify this claim, assume the contrary: there are z_1, z_2 in Z with $z_1 \prec z_2$, $z_2 \prec z_1$. By definition, there are y_1, y_2 in $X - Z$ such that $(y_1, z_1, z_2) \in \mathcal{B}$ and $(y_2, z_2, z_1) \in \mathcal{B}$. But then the hypothesis of the theorem is contradicted by $a = y_1$, $b = z_1$, $c_1 = y_2$, $c_2 = z_2$, $c_3 = z_1$: since $\tau_{\mathcal{N}}(X - Z) \cap Z = \emptyset$, we have $(y_1, z_1, y_2) \notin \mathcal{B}$.

In addition, we claim that

- $\prec$ is transitive.

To justify this claim, consider any z_1, z_2, z_3 in Z such that $z_1 \prec z_2$ and $z_2 \prec z_3$. By definition, there are y_1, y_2 in $X - Z$ such that $(y_1, z_1, z_2), (y_2, z_2, z_3) \in \mathcal{B}$; as $\tau_{\mathcal{N}}(X - Z) \cap Z = \emptyset$ guarantees that $(y_1, z_1, y_2) \notin \mathcal{B}$, the hypothesis of the theorem with $a = y_1$, $b = z_1$, $c_1 = y_2$, $c_2 = z_2$, $c_3 = z_3$ implies $(y_1, z_1, z_3) \in \mathcal{B}$, and so $z_1 \prec z_3$.

Our set Z is nonempty (it includes x_3) and it is partially ordered by $\prec$. Let $Z_{\max}$ denote the set of its maximal elements. We claim that

- $Z_{\max} \cap \mathrm{ex}_{\mathcal{B}}(Z) \subseteq \mathrm{ex}_{\mathcal{B}}(X)$.

To justify this claim, assume the contrary: there are a, z_2, c in X such that $z_2 \in Z_{\max} \cap \mathrm{ex}_{\mathcal{B}}(Z)$ and $(a, z_2, c) \in \mathcal{B}$. Since $\tau_{\mathcal{N}}(X - Z) \cap Z = \emptyset$, at least one of a, c belongs to Z; since $z_2 \in \mathrm{ex}_{\mathcal{B}}(Z)$, at most one of a, c belongs to Z; now symmetry allows us to assume that $a \in X - Z$ and $c \in Z$. But then $z_2 \prec c$, contradicting the assumption that $z_2 \in Z_{\max}$.

We shall complete the proof by showing that

- $Z_{\max} \cap \mathrm{ex}_{\mathcal{B}}(Z) \neq \emptyset$.

For this purpose, we rely on the induction hypothesis; note that $|Z| < |X|$ as Z includes neither x_1 nor x_2.

Case 1: $Z_{\max} \neq Z$.

In this case, let z be any maximal element of $Z - Z_{\max}$. Since $z \notin Z_{\max}$, there are a y in $X - Z$ and a z_2 in $Z_{\max}$ such that $(y, z, z_2) \in \mathcal{B}$. If $z_2 \in \mathrm{ex}_{\mathcal{B}}(Z)$, then we are done; else there are elements z_1, z_3 of Z such that $(z_1, z_2, z_3) \in \mathcal{B}$. Now the induction hypothesis applied to Z and (z_1, z_2, z_3) yields a $\overline{z}_3$ in $\mathrm{ex}_{\mathcal{B}}(Z)$ such that $(z_1, z_2, \overline{z}_3) \in \mathcal{B}$; next, the induction hypothesis applied to Z and $(\overline{z}_3, z_2, z_1)$ yields a $\overline{z}_1$ in $\mathrm{ex}_{\mathcal{B}}(Z)$ such that $(\overline{z}_3, z_2, \overline{z}_1) \in \mathcal{B}$. The hypothesis of the theorem with $a = y, b = z$, $c_1 = \overline{z}_1$, $c_2 = z_2$, $c_3 = \overline{z}_3$ guarantees that a subscript i in $\{1, 3\}$ satisfies $z \prec \overline{z}_i$; now maximality of z implies $\overline{z}_i \in Z_{\max}$.

Case 2: $Z_{\max} = Z$.

In this case, our task reduces to proving that $\mathrm{ex}_{\mathcal{B}}(Z) \neq \emptyset$. We may assume that $\mathrm{ex}_{\mathcal{B}}(Z) \neq Z$ (else we are done), and so $\mathcal{B}$ includes a triple (z_1, z_2, z_3) such that z_1, z_2, z_3 are elements of Z. But then the induction hypothesis applied to Z and (z_1, z_2, z_3) yields a $\overline{z}_3$ in $\mathrm{ex}_{\mathcal{B}}(Z)$. □

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