The Sylvester-Chvátal Theorem

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November 4, 2003

Abstract

The Sylvester-Gallai theorem asserts that every finite set S of points in two-dimensional Euclidean space includes two points, a and b, such that either there is no other point in S is on the line ab, or the line ab contains all the points in S. V. Chvátal extended the notion of *lines* to arbitrary metric spaces and made a conjecture that generalizes the Sylvester-Gallai theorem. In the present article we prove this conjecture.

1 Introduction

The following now-celebrated geometric problem was proposed by Sylvester in 1893 [12].

> *Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.*

In the twentieth century several proofs were found; the first one , published some fifty years after the problem was proposed and one year after Erdős revived it, was due to Gallai [9]. A very short proof was given by L.M. Kelly; this proof uses the notion of Euclidean distance and can be found in Coxeter ([4]; §4.7 of [5]) and Gale (Chapter 8 of [8]). Additional information on the Sylvester-Gallai theorem can be found in Borwein and Moser [1], Chvátal [3], Erdős and Purdy [7], and Pach and Agarwal [11].

In an arbitrary metric space (V, ρ), Menger [10] defined a ternary relation $\mathcal{B}(\rho)$ of *metric betweenness on* V by

$$(u, v, w) \in \mathcal{B} \ \Leftrightarrow u, v, w \ \text{are all distinct and} \ \rho(u, w) = \rho(u, v) + \rho(v, w). \tag{1}$$

We shall follow the tradition — adopted by Menger [10] — of writing $[uvw]$ for $(u, v, w) \in \mathcal{B}$. As Chvátal pointed out in [3], if we define the line ab as

$$\{x : [xab]\} \ \cup \ \{a\} \ \cup \ \{x : [axb]\} \ \cup \ \{b\} \ \cup \ \{x : [abx]\}, \tag{2}$$

*Supported by (theory of computation) NSF grant 0105692.

then a straightfoward generalization of the Sylvester-Gallai theorem in arbitary finite metric space is no longer true. Chvátal's [2] definition of a *line* in an arbitrary metric space is a recursive closure of (2); details are as follows.

First, the ternary relation $\mathcal{B}(\rho)$ in (1) is transformed into a set $\mathcal{H}(\mathcal{B}(\rho))$ of three-point subsets of V:

$$\mathcal{H}(\mathcal{B}(\rho)) = \{\{a, b, c\} : (a, b, c) \in \mathcal{B}(\rho)\}. \tag{3}$$

Then a family $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ of subsets of V is defined by

$$\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho))) = \{S \subseteq V : \text{ no } T \text{ in } \mathcal{H}(\mathcal{B}(\rho)) \text{ has } |T \cap S| = 2\}. \tag{4}$$

(It is clear that $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ is closed under arbitary intersection.) For any subset W of V, define $\mathcal{C}(W)$ as

$$\mathcal{C}(W) = \bigcap_{W \subseteq S \in \mathcal{A}} S. \tag{5}$$

Equivalently, if W is finite, then $\mathcal{C}(W)$ is the return value of the program

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S = W;
while some T in H(B(ρ)) has |T ∩ S| = 2 do S = S ∪ T end;
return S;
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Finally, for any two distinct points a and b of V, the *line ab* is defined as $\mathcal{C}(\{a, b\})$. With this definition, Chvátal made the following conjecture in [3], which is made into a theorem by the present paper.

Theorem 1 *If (V, ρ) is a metric space such that $1 < |V| < \infty$, then V contains two distinct points a and b such that the line ab is $\{a, b\}$ or V.*

The original Sylvester-Gallai Theorem is a special case of Theorem 1, where (V, ρ) is a finite subspace of the Euclidean plane.

For a finite, un-directed, connected graph $G = (V, E, w)$ with positive weights, there is naturally an induced metric space (V, ρ), where $\rho(x, y)$ is defined to be the distance between x and y in G. Note that every finite metric space is induced by such a graph. So it suffices to prove the conjecture for finite metric spaces induced by graphs. Although the proof does not need the notations in graphs, this was the setting where we worked on the conjecture.

Chvátal [3] proved this conjecture for all metric spaces with at most nine points and for all (finite) metric spaces induced by connected graphs with unit weights. Etin [2] proved the conjecture for all (finite) subspaces of ℓ_1^2, the two-dimensional space with the ℓ_1-metric. In prior work we developed techniques that allowed us to give a much simpler proof for the ℓ_1^2 case, as well as proofs for metric spaces induced by graphs with some special weights. Based on similar technique, just one week before we found the proof, Bin Tian proved the conjecture for all finite subspaces of ℓ_1^3 by using a computer program.

2 The Sylvester-Chvátal Theorem

We reserve the letter V for the ground-set of a finite metric space with at least two points and we reserve the letter ρ for the metric of this space. Our proof of Theorem 1 splits into two parts.

Theorem 2 *If every three points of V are contained in some line, then some line consists of all points of V.*

Theorem 3 *If some three points of V are contained in no line, then some line consists of precisely two points.*

Proof of Theorem 2. Consider a line, L, which is maximal with respect to set-inclusion; we claim that $L = V$. To justify this claim, assume the contrary: some point, c, of V lies outside L. Line L is generated by two points, a and b, of V. By assumption, a, b, c are contained in some line; this line contains $L \cup \{c\}$, contradicting maximality of L.

Proof of Theorem 3. By a *simple edge*, we mean any edge — ab — of the complete graph with vertex-set V such that no point x of V satisfies $[axb]$. By a *simple triangle*, we mean any three points — a, b, c — of V such that all of ab, bc, ca are simple edges. Now consider the following statements:

(i) some three points of V are contained in no line,
(ii) some simple triangle is contained in no line,
(iii) some line consists of precisely two points.

Proof of (i)⇒(ii). By (i), there are three points — a, b, c — of V such that

$$\text{no line contains } \{a, b, c\}; \tag{6}$$

among all such triples, choose one that minimizes $\rho(a, b) + \rho(b, c) + \rho(a, c)$; we claim that a, b, c is a simple triangle.

To justify this claim, assume the contrary: without loss of generality, there is a point d such that $[adb]$. Note first that $d \neq c$ (else (6) is contradicted by $[acb]$) and then that $\rho(d, c) < \rho(d, b) + \rho(b, c)$ (else (6) is contradicted by $[dbc]$ and $[adb]$). It follows that

$$\begin{aligned} & \rho(a, d) + \rho(d, c) + \rho(a, c) \\ < \;& \rho(a, d) + \rho(d, b) + \rho(b, c) + \rho(a, c) \\ = \;& \rho(a, b) + \rho(b, c) + \rho(a, c); \end{aligned}$$

now minimality of a, b, c implies that some line contains $\{a, d, c\}$; by $[adb]$, the same line contains $\{a, b, c\}$, contradicting (6).

Proof of (ii)⇒(iii). For each ordered triple u, v, w of points of V, let us write

$$\Delta(u, v, w) = \rho(u, v) + \rho(v, w) - \rho(u, w).$$

By (ii), some simple triangle — a, b, c — satisfies (6); among all such simple triangles, choose one that minimizes $\Delta(a, b, c)$; we claim that line $L(a, c)$ consists of precisely two points.

To justify this claim, assume the contrary: line $L(a, c)$ consists of at least three points. This means that some point d satisfies $[dac]$ or $[adc]$ or $[acd]$; since ac is a simple edge, $[adc]$ is excluded; now symmetry allows us to assume $[acd]$. Among all such points d, we choose one that minimizes $\rho(c, d)$; this property of d guarantees that cd is a simple edge. (Figure 1.)

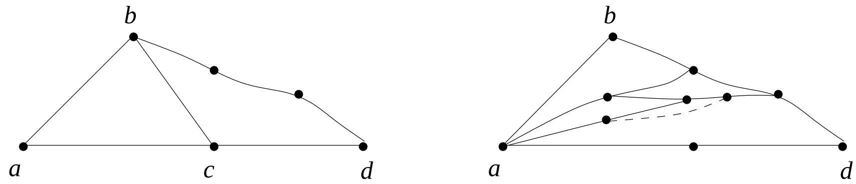

Figure 1: If line $L(a,c)$ consists of at least three points, the point d will cause infinite trouble.

Let us show that

$$bd \text{ is not a simple edge.} \tag{7}$$

If (7) is false, then (b,c,d) is a simple triangle; $[acd]$ and (6) guarantee that this simple triangle is contained in no line. It follows that $\Delta(b,c,d) \geq \Delta(a,b,c)$, which means $\rho(c,d) - \rho(b,d) \geq \rho(a,b) - \rho(a,c)$; since $[acd]$, we conclude $[abd]$; but then (6) is contradicted.

In addition, let us observe that

$$\rho(a,b) + \rho(b,d) < \rho(a,d) + \Delta(a,b,c) : \tag{8}$$

if (8) is false, then $[acd]$ guarantees $\rho(b,d) \geq \rho(b,c) + \rho(c,d)$, and so $[bcd]$; but then (6) is contradicted.

By a *path*, we mean any sequence — $(a_1, a_2, \dots, a_k)$ — of points of V; we define its *length* as

$$\textstyle\sum_{i=1}^{k-1} \rho(a_i, a_{i+1});$$

if the path is denoted P, then we denote its length $\ell(P)$. By a *special path*, we mean a path $(a_1, a_2, \dots, a_k)$ such that $a_1 = a$, $k \geq 3$, $a_k = d$,

(α) no line contains $\{a_1, a_2, a_3\}$, and

(β) at least one of a_1a_2 and a_2a_3 is not a simple edge.

Note that (a,b,d) is a special path: here, (α) follows from $[acd]$ with (6) and (β) follows from (7). Now we can choose a shortest special path, $(a_1, a_2, \dots, a_k)$; let us denote it P. Since (a,b,d) is a special path, (8) guarantees that

$$\ell(P) < \rho(a,d) + \Delta(a,b,c). \tag{9}$$

To complete the proof of (ii)$\Rightarrow$(iii), we shall distinguish between three cases.

Case 1: *Neither a_1a_2 nor a_2a_3 is a simple edge.*

By assumption of this case, there is a point b_{12} such that $[a_1b_{12}a_2]$. Among all such points b_{12}, we choose one that minimizes $\rho(b_{12}, a_2)$; this property of b_{12} guarantees that $b_{12}a_2$ is a simple edge. Similarly, there is a point b_{23} such that $[a_2b_{23}a_3]$ and such that a_2b_{23} is a simple edge. Note that

$$\text{no line contains } \{b_{12}, a_2, b_{23}\}: \tag{10}$$

else $[a_1b_{12}a_2]$, $[a_2b_{23}a_3]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{12}, b_{23}, a_3, \dots, a_k)$. From $[a_1b_{12}a_2]$ and $[a_2b_{23}a_3]$, we have

$$\ell(P) - \ell(P') = \Delta(b_{12}, a_2, b_{23});$$

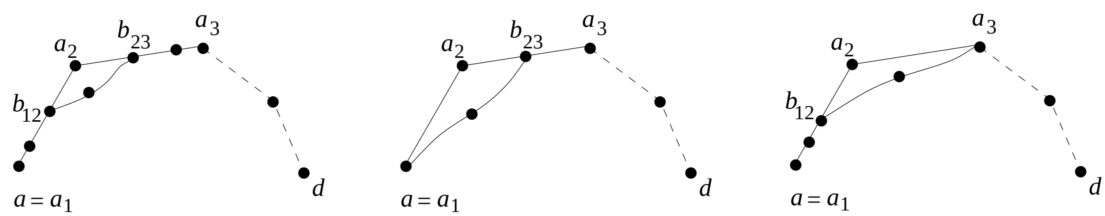

Figure 2: Three cases of the special path P.

(10) guarantees that $\Delta(b_{12}, a_2, b_{23}) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{12}, b_{23}\}$ (else $[a_1 b_{12} a_2]$ would guarantee that the same line contains $\{b_{12}, a_2, b_{23}\}$, contradicting (10)) and yet P' is not special, both $a_1 b_{12}$ and $b_{12} b_{23}$ are simple edges. Since $b_{12} b_{23}$ is a simple edge, b_{12}, a_2, b_{23} is a simple triangle, and so (10) implies $\Delta(b_{12}, a_2, b_{23}) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(b_{12}, a_2, b_{23}) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (9).

Case 2: *$a_1 a_2$ is a simple edge and $a_2 a_3$ is not.*

As in Case 1, there is a point b_{23} such that $[a_2 b_{23} a_3]$ and such that $a_2 b_{23}$ is a simple edge. Note that

$$\text{no line contains } \{a_1, a_2, b_{23}\}: \tag{11}$$

else $[a_2 b_{23} a_3]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{23}, a_3, \ldots, a_k)$. From $[a_2 b_{23} a_3]$, we have

$$\ell(P) - \ell(P') = \Delta(a_1, a_2, b_{23});$$

(11) guarantees that $\Delta(a_1, a_2, b_{23}) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{23}, a_3\}$ (else $[a_2 b_{23} a_3]$ would guarantee that the same line contains $\{a_1, a_2, b_{23}\}$, contradicting (11)) and yet P' is not special, both $a_1 b_{23}$ and $b_{23} a_3$ are simple edges. Since $a_1 b_{23}$ is a simple edge, a_1, a_2, b_{23} is a simple triangle, and so (11) implies $\Delta(a_1, a_2, b_{23}) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(a_1, a_2, b_{23}) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (9).

Case 3: *$a_2 a_3$ is a simple edge and $a_1 a_2$ is not.*

As in Case 1, there is a point b_{12} such that $[a_1 b_{12} a_2]$ and such that $b_{12} a_2$ is a simple edge. Note that

$$\text{no line contains } \{b_{12}, a_2, a_3\}: \tag{12}$$

else $[a_1 b_{12} a_2]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{12}, a_3, \ldots, a_k)$. From $[a_1 b_{12} a_2]$, we have

$$\ell(P) - \ell(P') = \Delta(b_{12}, a_2, a_3);$$

(12) guarantees that $\Delta(b_{12}, a_2, a_3) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{12}, a_3\}$ (else $[a_1 b_{12} a_2]$ would guarantee that the same line contains $\{b_{12}, a_2, a_3\}$, contradicting (12)) and yet P' is not special, both $a_1 b_{12}$ and $b_{12} a_3$ are simple edges. Since $b_{12} a_3$ is a simple edge, b_{12}, a_2, a_3 is a simple triangle, and so (12) implies $\Delta(b_{12}, a_2, a_3) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(b_{12}, a_2, a_3) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (9). □

3 An Application in Block Designs

Recall that a (v, k, λ) *design* is a hypergraph on a set V of v points with the property that any pair of two points is contained in exactly λ edges with k points in each edge. We say a block design on V is *realizable as a metric space* if there is a metric space (V, ρ) such that for any three points $a, b, c \in V$ we have

$$\{a, b, c\} \in \mathcal{H}(\mathcal{B}(\rho)) \text{ if and only if } \{a, b, c\} \text{ is contained in some edge of the design.}$$

Recall that a finite projective plane of order n is an $(n^2 + n + 1, n + 1, 1)$ design, a finite affine plane of order n is an $(n^2, n, 1)$ design, and a Steiner triple system is a $(v, 3, 1)$ design. If a $(v, k, 1)$ design is realizable as a metric space, then every line in the metric space contains exactly k points. Therefore, immediately following Theorem 1 we have

Corollary 4 *No $(v, k, 1)$ design with $k \geq 3$ and $v > k$ is realizable as a metric space. In particular, no projective plane of order higher than 1, nor any affine plane of order higher than 2, nor any Steiner triple system with more than 3 points is realizable as a metric space.*

Mario Szegedy, in a discussion on this subject, first asked the question whether there is a short proof for the fact that no projective plane of order higher than 1 is realizable as a metric space. To the best of the author's knowledge and ability, no simple proof is available; even the question is new to the literature. The special case of the Fano plane was solved by Chvátal in [3]; the proof was not simple.

Acknowledgments

I want to thank Vašek Chvátal for having introduced his nice conjecture to me. The way we present the proof of the main theorem here is also due to Vašek. In addition, I want to thank József Beck, Jeff Kahn, Bill Steiger, and Mario Szegedy for their helpful comments.

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