INTERSECTING FAMILIES OF EDGES

IN HYPERGRAPHS HAVING THE HEREDITARY PROPERTY

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I. Introduction. Let F be a hypergraph with vertex set

$$S = \{1, 2, \ldots, n\}$$

An intersecting family of edges in F is a partial hypergraph G such that

$$X, Y \in G \Rightarrow X \cap Y \neq \emptyset$$

If F is a simple graph, an intersecting family of edges is either a triangle or a star.

In a hypergraph F, the degree $\delta(x)$ of a vertex x is the number of edges containing x. Denote by

$$\delta(F) = \max_{x \in S} \delta(x)$$

the maximum degree in F. Clearly, the maximum size of an intersecting family is greater than or equal to $\delta(F)$.

Erdös, Chao-Ko and Rado have shown :

If F is complete r-uniform hypergraph with x vertices, $n \geq 2r$, then the maximum size of an intersecting family is equal to $\delta(F)$.

In this note, we use a similar technique to show that the same equality holds when the hypergraph F satisfies the following condition :

if $X_o \in F$, if $X \subseteq S$, and if there exists a one-to-one mapping f from X into X_o such that

$$f(x) \geq x \quad (x \in X),$$

then $X \in F$.

2. Let X , Y be sets of positive integers. If there is a one-to-one mapping $f: X \to Y$ with $x \leq f(x)$ for each $x \in X$ then we write $X < Y$. A family G of sets will be called intersecting if $X \cap Y \neq \emptyset$ whenever $X,Y \in G$.

Theorem. *Let F be a family of subsets of $\{1,2,\ldots,n\}$ such that $X \in F$, $Y < X \Rightarrow Y \in F$. Let G be an arbitrary intersecting subfamily of F.* Then

$$|G| \leq |\{X \in F: 1 \in X\}| \quad . \tag{1}$$

Proof. We will proceed by induction on n ; the case $n = 1$ is trivial. Now, let n be greater than one and let F , G satisfy the hypothesis of our theorem. To each family F^* of subsets of $\{1,2,\ldots,n\}$, we assign a weight $w(F^*) = \sum\sum k$ where the first sum runs over all $X \in F^*$ and the second one over all $k \in X$. Since we are going to prove (1), only the cardinality of G is of interest to us. Hence we may assume, without loss of generality, that G minimizes the weight among all the intersecting subfamilies of F having $|G|$ sets. First of all, we will prove that

$$X \in G\,,\ t \in X\,,\ s \notin X\,,\ s < t \ \Rightarrow\ (X - \{t\}) \cup \{s\} \in G \quad . \tag{2}$$

For this purpose, we will use the technique developed in [1]. Assume the contrary, i.e., let there be X , s , t violating (2). Fix s , t and set

$$G^* = \{Y \in G:\ t \in Y\,,\ s \notin Y\,,\ (Y - \{t\}) \cup \{s\} \notin G\} \quad .$$

Then $X \in G^*$. Moreover, let us set

$$H^* = \{(Y-\{t\}) \cup \{s\}: Y \in G^*\} \quad ,$$

$$H = H^* \cup (G - G^*) \quad .$$

Obviously, $|H| = |G|$, $H \subset F$ and $w(H) < w(G)$. By the minimality of $w(G)$, the family H cannot be intersecting. Since H^* and $G-G^*$ are both intersecting, there must be disjoint sets $Y \in H^*$ and $Z \in G-G^*$. Since $s \in Y$, we have $s \notin Z$. But $(Y-\{s\}) \cup \{t\} \in G^*$ and so $((Y-\{s\}) \cup \{t\}) \cap Z \neq \emptyset$. Therefore necessarily $t \in Z$. Since $Z \notin G^*$, we have $(Z-\{t\}) \cup \{s\} \in G$. Hence

$$\emptyset \neq ((Z-\{t\}) \cup \{s\}) \cap ((Y-\{s\}) \cup \{t\}) = (Y \cap Z) - \{s,t\}$$

contradicting $Y \cap Z = \emptyset$. Thus (2) is proved.

Next, let us note that, for any subsets X , Y of $\{1,2,\ldots,n\}$, $Y < X$ holds if and only if

$$|Y \cap \{k,k+1,\ldots,n\}| \leq |X \cap \{k,k+1,\ldots,n\}| \qquad (1 \leq k \leq n) \ .$$

Therefore

$$Y < X \iff \{1,2,\ldots,n\}-X < \{1,2,\ldots,n\}-Y \quad . \tag{3}$$

Let us set

$$F_1 = \{X \in F: \{1,2,\ldots,n\}-X \in F\} \quad .$$

From (3), we easily deduce that

$$X \in F-F_1 , Y < X \Rightarrow Y \in F-F_1 \quad . \tag{4}$$

Indeed, $X \in F$ and $Y < X$ imply $Y \in F$. If $Y \notin F-F_1$ then necessarily $Y \in F_1$, i.e., $\{1,2,\ldots,n\}-Y \in F$. By (3), we then have $\{1,2,\ldots,n\}-X \in F$ contradicting $X \notin F_1$.

Now, set

$$F_2 = \{X \in F - F_1 : n \notin X\} \quad ,$$

$$F_3 = \{X \in F - F_1 : n \in X\} \quad ,$$

$$F_3^* = \{X - \{n\} : X \in F_3\} \quad .$$

From (4), it follows easily that

$$X \in F_2 , Y < X \Rightarrow Y \in F_2 \quad ,$$

$$X \in F_3^* , Y < X \Rightarrow Y \in F_3^* \quad .$$

We also set

$$G_i = G \cap F_i \qquad (i = 1,2,3) \quad ,$$

$$G_3^* = \{X - \{n\} : X \in G_3\} \quad .$$

and finally, let us set

$$H = \{X \in F : 1 \in X\} \quad ,$$

$$H_i = H \cap F_i \qquad (i = 1,2,3) \quad ,$$

$$H_3^* = \{X - \{n\} : X \in H_3\} \quad .$$

If $Y, Z \in G_3$ then $Y \cup Z \neq \{1,2,\ldots,n\}$ (otherwise $Y, Z \in F_1$). Therefore there is a $k \in \{1,2,\ldots,n-1\}$ with $k \notin Y$, $k \notin Z$. By (2), one has $(Y-\{n\}) \cup \{k\} \in G$ and so

$$(Y-\{n\}) \cap (Z-\{n\}) = ((Y-\{n\}) \cup \{k\}) \cup Z \neq \emptyset \quad .$$

Hence G_3^* is an intersecting subfamily of F_3^* .

Now, we can apply the induction step, obtaining thus

$$|G_2| \leq |H_2| \tag{5}$$

and

$$|G_3| = |G_3^*| \leq |H_3^*| = |H_3| \quad . \tag{6}$$

Finally, it is easy to see that

$$|G_1| \leq \frac{1}{2}|F_1| = |H_1| \quad . \tag{7}$$

Indeed, the family F_1 can be split into pairs $(X, \{1,2,\ldots,n\}-X)$. At most one element of each pair can be included in G_1 ; exactly one element of each pair is included in H_1 .

Summing up (5), (6) and (7) we obtain

$$|G| = |G_1| + |G_2| + |G_3| \leq |H_1| + |H_2| + |H_3| = |H|$$

which is the desired result. The proof is finished.

Perhaps the following strengthening of our theorem still remains valid:

Conjecture. Let F be a family of subsets of a finite set S such that $X \in F, Y \subset X \Rightarrow Y \in F$. Then there is a $t \in S$ such that every intersecting subfamily G of F satisfies

$$|G| \leq |\{X \in F : t \in X\}| \quad .$$

One might also believe that the following generalization of our theorem is true: Let F be a family of subsets of $\{1,2,\ldots,n\}$ such that $X \in F, Y < X \Rightarrow Y \in F$; let G be a subfamily of F containing no $k+1$ pairwise disjoint sets and such that $|G| > k$. Then

$$|G| \leq |\{X \in F: \{1,2,\ldots,k\} \cap X \neq \emptyset\}| \quad . \tag{8}$$

However, this statement is false whenever $k > 1$. Indeed, if F consists of all the subsets of $\{1,2,\ldots,2k+1\}$ then the right-hand side of (8) is $2^{2k+1}-2^{k+1}$. However, the family

$$G = \{X \subset \{1,2,\ldots,2k+1\}: |X| \geq 2\}$$

has no $k+1$ pairwise disjoint sets and includes

$$2^{2k+1} - (2k+2) > 2^{2k+1} - 2^{k+1}$$

sets. Nevertheless, it would be desirable to prove (8) under more restrictive conditions on F . Such a theorem might eventually imply the following number-theoretical conjecture of Erdös: Let S be a subset of $\{1,2,\ldots,m\}$ containing no $k+1$ pairwise coprime integers. Then $|S| \leq |T|$ where T is obtained by taking all those integers in $\{1,2,\ldots,m\}$ which are multiples of (at least one of) the first k primes.

REFERENCES

1. P. Erdös, Chao-Ko and R. Rado, "Intersection theorems for systems of finite sets," Quarterly J. of Math. (Oxford, 2nd sec.) 12 (1961), 313-320.