A De Bruijn–Erdős theorem in graphs?

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Abstract A set of n points in the Euclidean plane determines at least n distinct lines unless these n points are collinear. In 2006, Chen and Chvátal asked whether the same statement holds true in general metric spaces, where the line determined by points x and y is defined as the set consisting of x, y, and all points z such that one of the three points x, y, z lies between the other two. The conjecture that it does hold true remains unresolved even in the special case where the metric space arises from a connected undirected graph with unit lengths assigned to edges. We trace its curriculum vitae and point out twenty-nine related open problems plus three additional conjectures.

1 Prehistory

It all started when Alain Guenoche and Bernard Fichet asked me if I wanted to come to their third International Conference on Discrete Metric Spaces in Marseilles in September 1998. I like metric spaces and I love Marseilles, I replied, but I have no results I could present there. Never mind, they said magnanimously, come anyway. As I am not completely without shame, I then began racking my brain for something to talk about at the conference. A distant memory came to the rescue: As an undergraduate, I marvelled at the interpretation of families of sets as metric spaces provided by the Hamming metric on a family of indicator functions. Could a few combinatorial theorems be generalized to the realm of metric spaces? Dusting off my youthful ambition thirty years later, I circled around it till I settled on the project of looking for theorems of Euclidean geometry that might be generalized to arbitrary metric spaces.

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1.1 Lines and closure lines in metric spaces

Saying that point v in a Euclidean space lies between points u and w means that v is an interior point of the line segment with endpoints u and w; line $\overline{xy}$ is the set consisting of x, y, and all points z such that one of the three points x,y,z lies between the other two. These notions have straightforward extensions to arbitrary metric spaces: In a space with metric $dist$, saying that point v *lies between* points u and w means that u,v,w are pairwise distinct and $dist(u,v)+dist(v,w) = dist(u,w)$; if *line* $L(xy)$ is defined as the set consisting of x, y, and all points z such that one of the three points x,y,z lies between the other two, then $L(xy) = \overline{xy}$ in the special case where $dist$ is the Euclidean metric.

This was the definition of lines in metric spaces that I hoped to use in extending a theorem or two of Euclidean geometry to arbitrary metric spaces. One candidate was the Sylvester–Gallai theorem [34, 22],

> Every non-collinear finite subset V of the Euclidean plane such that $|V| \geq 2$ includes two points such that the line determined by them passes through no other point of V,

whose generalization would read

> In every finite metric space $(V,dist)$ such that $|V| \geq 2$, some line consists of only two points of V or of all points of V.

This candidate flunked miserably: When $(V,dist)$ is the pentagon C_5 with the usual graph metric (in this case, $dist(x,y) = 1$ when vertices x,y are adjacent and $dist(x,y) = 2$ when vertices x,y are nonadjacent), $L(xy)$ consists of four vertices when x,y are adjacent and it consists of three vertices when x,y are nonadjacent.

Undaunted, I tried another tack: Let us define *closure line* $C(xy)$ as the smallest superset of $L(xy)$ such that $u,v \in C(xy) \Rightarrow L(uv) \subseteq C(xy)$. Just like lines $L(xy)$, closure lines $C(xy)$ are identical with Euclidean lines $\overline{xy}$ in the special case where the metric is Euclidean. Unlike lines $L(xy)$, closure lines $C(xy)$ did not flunk the Sylvester–Gallai test at once: I could not find a counterexample to the statement

> (SG) In every finite metric space $(V,dist)$ such that $|V| \geq 2$, some closure line consists of only two points of V or of all points of V.

(In particular, C_5 is not a counterexample as each of its ten closure lines consists of all five vertices.)

1.2 Sylvester–Gallai theorem in metric spaces?

Having formulated generalization (SG) of the Sylvester–Gallai theorem, I tried to prove it. The known proofs of the Sylvester–Gallai theorem [21, 17, 18] did not help: I failed to adapt any of them to a proof of (SG). I considered the restricted version of (SG) where the metric spaces are induced by graphs: every connected undirected graph with vertex set V induces the metric space $(V, dist)$ where $dist$ is the usual graph metric ($dist(u,v)$ standing for the number of edges in the shortest path from u to v). This turned out to be easy: not only the restricted version of (SG), but even a stronger statement,

> In every finite metric space $(V, dist)$ induced by a graph with at least two vertices, every closure line consists of only two points of V or of all points of V,

is valid. (The proof is a simple exercise: if x and y are adjacent twins, then $C(xy) = \{x,y\}$; else $C(xy) = V$.) To get more faith in the validity of (SG), I then tried to show that a counterexample could not be ridiculously small; plodding case analysis aided by computer search established that (SG) holds true for all metric spaces with at most nine points. Armed with this pathetic evidence, I presented the arrogant conjecture and related observations [15] at the Marseilles meeting.

Over the next few years, I publicized the conjecture vigorously. I told it to anybody who would listen. I told it to first-class researchers and some of them may have taken a crack at it. I gave talks about it in different places. A mathematical luminary interrupted my lecture at Princeton to announce that he had a counterexample; a few minutes later he and the entire audience agreed that the example was not a counterexample. After the lecture; he proposed to me (now privately) a new counterexample; this, too, turned out to be false. Such episodes made me feel that the conjecture may have been not all that arrogant.

The conjecture remained unresolved till the fall of 2003.

1.3 Enter Xiaomin

It was March 2000. There was a knock and when I opened my office door, there stood a young man who asked for a few minutes of my time. He explained to me his personal reasons for wanting to come to Rutgers as a graduate student in the middle of spring term and asked me if I could help by putting in a good word for him.

I said I sympathized, but as I didn't know him from Adam, I could not put in a good word for him. He replied that he anticipated this reaction and perhaps I could give him a test to get an idea of his mathematical abilities? As I was just about to leave for my graduate class in algorithms and data structures, I handed to him a copy of the midterm exam I was going to give in a few minutes and asked him to come back after class. He looked the exam over, asked for definitions of a couple of concepts he was unfamiliar with, and then we went our separate ways. When I returned and read his answers, my jaw dropped: they were a notch above those of the thirty students who had studied the material for half a term. It was only later and after much prodding from me that he reluctantly confessed to his high ranking in the Chinese Mathematical Olympiad. (China being a biggish country, I was much impressed, of course.)

I gave him a glowing recommendation, he was admitted, and the rest is history. His name was Xiaomin Chen.

1.4 Sylvester–Gallai theorem in metric spaces!

In the fall of 2003, Xiaomin proved conjecture (SG).

A pivotal notion in Leroy Milton Kelly's celebrated short proof [17], [18, Section 4.7], [24, Chapter 8] of the Sylvester–Gallai theorem is the distance of a point from a line. This notion is unavailable in general metric spaces and yet echoes of Kelly's proof can be found in Chen's. Kelly minimizes the distance of point b from line $\overline{ac}$ over all noncollinear triples a,b,c, which can be seen as choosing the flattest triangle with base ac and apex b; Chen minimizes $dist(a,b)+dist(b,c)-dist(a,c)$, which can also be seen as choosing the flattest triangle with base ac and apex b. Here, the following definitions are required to overcome a technical wrinkle: in a metric space:

- a *triangle* is a set of three points, none of which lies between the other two;
- its three *edges* are its two-point subsets;
- an edge is *simple* if no point lies between its two points;
- a triangle is *simple* if all three of its edges are simple.

Synopses of the two proofs are compared in Table 1.

Table 1 Comparison of the two proofs

Euclidean plane: Kelly	General metric space: Chen
1. *If some three points of V are noncollinear, then some line passes through only two points of V:*	1A. *If some three points are in no closure line, then some simple triangle is in no closure line:*
	if a,b,c minimize $dist(a,b)+dist(b,c)+dist(a,c)$ over all triples of points in no closure line, then $\{a,b,c\}$ is a simple triangle.
	1B. *If some simple triangle is in no closure line, then some closure line consists of two points:*
if a,b,c minimize the distance of point b from line $\overline{ac}$, over all noncollinear triples, then $\overline{ac}$ passes through no third point of V.	if a,b,c minimize $dist(a,b)+dist(b,c)-dist(a,c)$ over all simple triangles, then $C(ac)=\{a,c\}$.
2. *If every three points of V are collinear, then all points of V are collinear.*	2. *If every three points are in some closure line, then some closure line consists of all the points.*

As for the devil in the details, the first part of Kelly's proof is crisp: if $\overline{ac}$ included three points x,y,z with y between x and z, then the distance of y from $\overline{bx}$ or the distance of y from $\overline{bz}$ would be smaller than the distance of b from $\overline{ac}$, a contradiction. By contrast, part 1B of Chen's proof is far from straightforward.

2 A De Bruijn–Erdős theorem in metric spaces?

When I was publicizing the conjecture that the Sylvester–Gallai theorem extends to metric spaces, Victor Klee and Benny Sudakov (independently of each other) suggested to me that other theorems on points and lines in the Euclidean plane might be eligible for a similar treatment. One of these is another well-known theorem,

Theorem 1. *Every non-collinear finite subset V of the Euclidean plane such that $|V| \geq 2$ determines at least $|V|$ distinct lines.*

As Paul Erdős [21] remarked in 1943, Theorem 1 follows easily by induction from the Sylvester–Gallai theorem:

> A line passing through only two points of V, point x and another one, does not belong to the set of lines determined by $V - \{x\}$. If this set includes at least $|V| - 1$ distinct lines, then V determines at least $|V|$ distinct lines; else, by the induction hypothesis, $V - \{x\}$ is collinear, in which case lines $\overline{xy}$ with y ranging over $V - \{x\}$ are pairwise distinct.

In 2006, Xiaomin and I set out to investigate whether Theorem 1 could be generalized to metric spaces. Candidate

> In every finite metric space $(V, dist)$ with at least two points, there are at least $|V|$ distinct closure lines or some closure line consists of all points of V

for such a generalization flunked badly:

Theorem 2. [11, Theorem 7] *For every integer n greater than 5, there is a metric space on n points where there are precisely 7 distinct closure lines and each closure line consists of at most $n - 2$ points.*

Nevertheless, we could not find a counterexample with 'closure lines' replaced by 'lines'.

> *Conjecture 1.* In every finite metric space with n points such that $n \geq 2$, there are at least n distinct lines or some line consists of all n points.

2.1 Terminology

2.1.1 Two De Bruijn–Erdős theorems and one that is not

Two joint results of Nicolaas Govert de Bruijn and Paul Erdős share the name 'De Bruijn–Erdős theorem':

Theorem 3. [19] *Let m and n be positive integers such that $m \geq 2$; let V be a set of n points; let E be a family of m subsets of V such that every two distinct points of V belong to precisely one member of E. Then $m \geq n$, with equality if and only if*

- *$\mathscr{L}$ is of the type $\{p_1, p_2, \dots, p_{n-1}\}$, $\{p_1, p_n\}$, $\{p_2, p_n\}$, $\dots$, $\{p_{n-1}, p_n\}$*

or

- *$n = k(k-1)+1$ with each member of $\mathscr{L}$ containing k points of V and each point of V contained in k members of $\mathscr{L}$.*

Theorem 4. [20] *Let k be a positive integer and let G be an infinite graph. If every finite subgraph of G is k-colourable, then G is k-colourable.*

Theorem 1 is sometimes incorrectly referred to as the 'De Bruijn–Erdős theorem'. The confusion has no doubt originated from the fact that it is a special case of the far more powerful Theorem 3. I apologize for having been one of the culprits perpetuating this error [11, 13] and I plead initial ignorance.

So how should we refer to the proposition in Conjecture 1? On the one hand, Theorem 1 has no name; on the other hand, 'De Bruijn–Erdős theorem in metric spaces', while incorrect, is crisp and has been used for years. Let us stick to it.

2.1.2 What is the meaning of 'conjecture'?

For some people, conjecturing X seems to imply commitment to the belief that X is true. (My doctoral adviser Crispin St. John Alvah Nash-Williams would use the term 'possible conjecture' when he was in doubt.) I am not one of these people: to me, conjecturing X means that (i) I am interested in the truth value of X and (ii) I have no counterexample. Still, when Xiaomin and I were writing our paper [11], I phrased the problem as a question rather than a conjecture. I do not recall my reason for this cowardice. Anyway, what's in a name? That which we call a conjecture by any other name would … [33].

2.2 A logbook

In the year 2000, the Government of Canada created a permanent program to establish 2000 research professorships called *Canada Research Chairs* in eligible degree-granting institutions across the country. I was privileged to hold first the Canada Research Chair in Combinatorial Optimization since my coming to Concordia in 2004 till 2011 and then the Canada Research Chair in Discrete Mathematics from 2011 till my retirement in 2014. Generous support from the program helped me in attracting brilliant postdocs and stellar visitors.

We worked and played in the framework of a research group that met weekly for problem-solving sessions. To conform to the fashion of naming everything by an acronym, I dubbed it *ConCoCO* for Concordia Computational Combinatorial Optimization. ConCoCO became a meeting ground not only for my students and postdocs, but also for students and faculty from other Montreal universities and for short-time visitors. I was particularly touched by the steady an enthusiastic support of Luc Devroye, my long-time friend and former colleague at McGill.

How is all this relevant to the subject at hand? The range of topics discussed at ConCoCO extended far beyond Conjecture 1, but the conjecture was its important part and the early results related to it came from ConCoCO.

- On 1 October 2006. Xiaomin and I first publicized our conjecture (oops! a question) along with a proof that every metric space on n points has at least $\lg n$ distinct lines or a line consisting of all n points.

- On 2 October 2006, ConCoCO was inaugurated.

- In September 2007, Ehsan Chiniforooshan joined ConCoCO as its first postdoc. He quickly became its cornerstone and then stayed on for two years. Two of our joint results (written up in May 2009) state that

  – [13, Corollary 1] in every metric space induced by a connected graph on n vertices, there are $\Omega(n^{2/7})$ distinct lines or else some line consists of all n vertices,
  – [13, Theorem 3] in every metric space on n points where each nonzero distance equals 1 or 2, there are $\Omega(n^{4/3})$ distinct lines and this bound is tight.

  The lower bound $\Omega(n^{2/7})$ was later improved to $\Omega(n^{4/7})$: see Subsection 3.2. The lower bound $\Omega(n^{4/3})$ was later extended to metric spaces where each nonzero distance equals 1, 2, or 3: see Subsection 4.1.

- In August 2010, Google Scholar surprised me by telling me that we were not alone in the universe: others were interested in Conjecture 1, too. I cannot resist the temptation to quote the opening of [27] verbatim (except for the reference labels):

  > In this paper, we present several results motivated by an open problem presented by Chvátal in the problem session of IWOCA 2008. We study systems of lines in metric spaces induced by graphs. Lines considered in this paper are sets of vertices defined by a relation of betweenness, as introduced by Menger [32]. A line containing all the vertices is called a universal line. Similar properties, concerning distances in graphs, are studied in metric graph theory, see a survey by Bandelt and Chepoi [6].
  >
  > The problem presented by Chvátal at IWOCA 2008 was originally conjectured by Klee and Wagon [29]. It is a generalization of the De Bruijn–Erdős Theorem [19]. The conjecture states the following:
  >
  > *Every graph with n vertices defines at least n different lines or it contains a universal line.*
  >
  > Klee and Wagon even stated this question about general discrete metric spaces, but we consider only graphs. This problem is still open, see [11].

  The reader will draw her own conclusions.

- In January 2011 I wrote to the legendary Maria Chudnovsky

> Some time during the period March – September of this year, six of us are going to get together in Montreal for a week or so for a concentrated attack on a couple of problems concerning lines in finite metric spaces. The other five are strong mathematicians, but we have not made too much progress so far and I am beginning to wonder if the two problems are not beyond our reach.
>
> And then it occurred to me that if only we could get you interested, you would end our misery by either solving the problems or, in the other case, certifying that they really are difficult.
>
> So I wonder if there is any way to persuade you to join us. I would pay for your travel and hotel, of course, plus a little extra as a Concordia's visitor. And we could schedule the workshop (not concurrent with but) adjacent to any event of your choice, such as the jazz festival or the film festival or anything else that you can find at
> `http://www.tourisme-montreal.org/What-To-Do/Events`

To my great delight, she accepted this invitation.

Our meeting took place between June 3 and June 13. Eight of us were at its center: Laurent Beaudou (ConCoCO participant from April to November 2008 and then its postdoc from September 2009 to March 2010), Adrian Bondy (my longtime friend and favourite collaborator), Xiaomin, Ehsan, Maria, Nico Fraiman (ConCoCO participant since January 2011), Yori Zwols (Maria's multiple coauthor and former doctoral student), and myself.

At our disposal we had the cozy meeting room EV3.101 on the third floor of Concordia's Engineering, Computer Science, and Visual Arts Complex with its large windows overlooking rue St. Catherine Street (as the signs in our bilingual Montreal used to say). There we would congregate every day from Saturday to Saturday at the crack of dawn and half an hour or so later, around 11AM, get down to business. In gruelling and greatly gratifying jam sessions we worked through the day nonstop, except for a short lunch break and a longer dinner break. (An important part of my job as the organizer was proposing new lunch and dinner venues every day.) These jam sessions went on till late at night and emails were flying from hotel room to hotel room well past midnight.

I brought two problems to the workshop. The first was to prove Conjecture 1 for metric spaces arising from connected chordal graphs and the second was to prove a generalization of the real De Bruijn–Erdős Theorem 3 in terms of 3-uniform hypergraphs (see Subsection 4.3). We solved the first problem as a warm-up in a day and half; the result was published much later as a note [8] contending for the record of the least number of lines per author. The second problem turned out to be more difficult; it led us on an emotional roller coaster where exhilirating victories were rapidly turning into crushing defeats. By Friday evening we were sure of a consolation prize, a weaker and less elegant version of what I had proposed. Saturday morning we luxuriated in a proof of the whole thing. Saturday afternoon revealed a big hole in this proof, in the night from Saturday to Sunday the roller coaster rolled on, and on Sunday morning the hole was patched up. The workshop ended up on this fairy-tale note and the result was published some thirty months later [7].

- Eleven weeks later, Yori joined ConCoCO as a postdoc for 2011–2012.

- In January 2012, Cathryn Supko defected from McGill in order to become my M.Comp.Sc. student. She participated in ConCoCO with remarkable energy until her graduation in July 2014.

- In April 2012, getting ready for the session ”My Favorite Graph Theory Conjectures” of the June 2012 SIAM Conference on Discrete Mathematics in Halifax, I was reminded of a three years old irritant: Ehsan and I had proved that n-point metric spaces where each nonzero distance equals 1 or 2 have $\Omega(n^{4/3})$ distinct lines, but we had not quite proved Conjecture 1 for these metric spaces. Even though our result provides a lower bound that is asymptotically far stronger than what the conjecture requires, it implies only that counterexamples to the conjecture, if any, include only finitely many metric spaces where each nonzero distance equals 1 or 2. I set out to remove this blemish. As it turned out, a variation on the arguments used in [13] proved that the smallest counterexample to the restricted conjecture had to have at most 7 points and then plodding case analysis took care of the rest. Ehsan was clearly entitled to a joint authorship, but he thought otherwise, and so I publicized it in `arXiv`, and published it twenty-eight months later [16], as a single author.

- In the same month, I found once again that interest in Conjecture 1 was not confined to our private group. This time, the outsiders' contribution was serious: Ida Kantor announced her talk at the forthcoming SIAM Conference on Discrete Mathematics. The talk presented results of her joint work with Balázs Patkós on the conjecture restricted to the plane with the ℓ_1 metric (Theorems 17 and 18 in Subsection 4.1). Their admirable achievements made us feel less incestuous and their proof techniques inspired further work on the conjecture [3].

- In the spring of 2013, I accepted two excellent postdocs, Pierre Aboulker and Rohan Kapadia, for the next academic year. (Later on, Rohan extended his stay by another year.) Around this time, Adrian told me of his planned visit to Montreal in late April and I jumped at the opportunity to try and re-create the magic atmosphere of July 2011.

  The new workshop took place on April 14 – 27. Heraclitus was right, no man ever steps in the same river twice. Heavy teaching schedule prevented Maria from coming and Yori could not get away from his commitments, either. Ehsan, having kept in touch with us by email during the first week, drove to Montreal on Friday the 19th and back to Waterloo on Wednesday the 24th. Xiaomin was in constant touch with us by email from Shanghai. Pierre, the fresh PhD, joined us for the duration; Rohan, just before his own defense, could not. Cathryn, Laurent, Nico (and, of course, Adrian) were present all the time.

  Our aim was to improve the lower bound $\lg n$ on the number of distinct lines in n-vertex 3-uniform hypergraphs where no line consists of all n vertices [11, Theorem 4]; see Subsection 4.3 for the definition. We kept improving the coefficient 1 in front of the $\lg n$ little by little until Xiaomin's brilliant friend Peihan Miao, then a junior student in Shanghai Jiaotong University (and now a doctoral student at Berkeley) pushed it all the way up to $2-o(1)$. We published this result in [2].

- In November 2013, Pierre and Rohan proved Conjecture 1 for metric spaces arising from connected distance-hereditary graphs [1].

- In the spring of 2014, Guangda Huzhang studied in his undergraduate thesis at Shanghai Jiaotong University geometric dominant metric spaces and graphs. His work was later expanded into a paper written jointly with Xiaomin, Peihan, and another of their friends, Kuan Yang [12]. Some of their results are quoted here in Subsection 3.5.

- ConCoCO held its last meeting on Thursday, 26 June 2014.

- With my retirement on August 31, 2014, research into Conjecture 1 gained a new momentum:

  - In the final four months of 2014, Pierre, Xiaomin, Guangda, Rohan, and Cathryn completed a project they had been working on since the beginning of the year. They published its results in [3]. Some of them are quoted here as Theorem 5 in Subsection 3.2, Theorem 19 in Subsection 4.1, and Theorem 25 in Subsection 4.2.

  - On 28 January 2015, Pierre and his friends Guillaume Lagarde, David Malec, Abhishek Methuku, and Casey Tompkins posted on `arXiv` their manuscript that was later published as [4]. Their results involve an analog of Conjecture 1 for a class of 3-uniform hypergraphs, which is quoted here as Theorem 28 in Subsection 4.3.

  - On 20 June, 2016, Pierre and his friends Martin Matamala, Paul Rochet, and José Zamora posted on `arXiv` their manuscript that was later published as [5]. They proved Conjecture 1 for metric spaces arising from a class of graphs that contains all connected chordal graphs and all connected distance-hereditary graphs. In addition, they proposed an intriguing variation on Conjecture 1. Some of their results are quoted here in Subsection 3.7.

3 A De Bruijn–Erdős theorem in graphs?

Every connected undirected graph G gives rise to the metric space $M(G)$ by defining the distance between two vertices as the smallest number of edges in a path joining them. Let us not play at being pedants: Rather than talking of lines in $M(G)$, let us talk of lines in G. Conjecture 1 remains open even in the special case where the metric space arises from a graph (and isn't this fortunate, since otherwise how could I submit this piece to the collection entitled '*Graph Theory* Favorite Conjectures and Open Problems'?).

Conjecture 2. In every finite connected graph with n vertices such that $n \geq 2$, there are at least n distinct lines or some line consists of all n vertices.

3.1 Terminology and notation

All our graphs as well as metric spaces and related objects are finite (unless specified otherwise), and so we will skip the qualifier 'finite' throughout the text. All our graphs are also undirected and connected (unless specified otherwise), and so we will skip these two qualifiers as well. To avoid the one-vertex graph (which has no lines at all), let us also agree that all our graphs have at least two vertices.

We let $|G|$ denote the number of vertices in a graph G. A line in a graph G is said to be *universal* if it consists of all $|G|$ vertices. A graph G is said to have *the DBE property* if it has at least $|G|$ distinct lines or a universal line. In these terms and under our assumptions, Conjecture 2 asserts that all graphs have the DBE property.

Sometimes we write simply uv for the unordered pair $\{u,v\}$ of distinct elements u and v.

3.2 A weaker lower bound attained by all graphs

Proving that almost all graphs have $\Theta(n^2)$ distinct lines, whether they have a universal line or not, is an easy exercise. This is far more than Conjecture 2 requires. When it comes to all graphs, we have only far less than Conjecture 2 requires.

Theorem 5. [3, Theorem 7.6] *Every graph G has $\Omega(|G|^{4/7})$ distinct lines or a universal line.*

3.3 Special cases where the lower bound is attained

One way of making progress toward the proof of Conjecture 2 is finding larger and larger classes of graphs with the DBE property. By now, we know three such classes:

- **Theorem 6.** *All bipartite graphs have the DBE property.*

  This theorem is just a simple observation: in a bipartite graph, $L(uv)$ is universal whenever u and v are adjacent. (For every vertex w, we have $|dist(w,u) - dist(w,v)| \leq 1$ and, since the graph is bipartite, $dist(w,u) \neq dist(w,v)$.) For its strengthening, see Theorem 16 in Subsection 3.7.

- **Theorem 7.** [16, special case of Theorem 1] *All graphs of diameter* 2 *have the DBE property.*

- **Theorem 8.** [5, corollary of Theorem 2.1] *All graphs that can be constructed from chordal graphs by repeated substitutions and gluing vertices have the DBE property.*

  Theorem 8 provides a common generalization of two previous results,

  – *all chordal graphs have the DBE property* [8, Theorem 1] and
  – *all distance-hereditary graphs have the DBE property* [1, Theorem 1].

  For its strengthening, see Theorem 15 in Subsection 3.7.

Here are three challenges motivated by these three theorems:

A graph is called *bisplit* [9] if its vertex set can be partitioned into stable sets X, Y, and Z so that $Y \cup Z$ induces a complete bipartite graph. (Bipartite graphs are bisplit graphs with $Z = \emptyset$.)

Problem 1. Prove that all bisplit graphs have the DBE property.

Problem 2. Prove that all graphs of diameter 3 have the DBE property.

It is known that every graph G of diameter 3 has at least $|G|/15$ distinct lines or a universal line: more generally, every graph G of diameter k has at least $|G|/5k$ distinct lines or a universal line (see Theorem 23 in Subsection 4.1.)

The *house* is the complement of the chordless path on five vertices; a *hole* is a chordless cycle with at least five vertices; the *domino* is the cycle on six vertices with one long and no short chord. An *HHD-free graph* [26] contains no house, no hole, and no domino as an induced subgraph. All graphs featured in Theorem 8 are HHD-free, but not all HHD-free graphs can be constructed as in Theorem 8. For example, start with the C_4 that has vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1$. Then, for each of the two $i = 1, 2$, substitute a clique $\{a_i, c_i, e_i\}$ for v_i, add vertices b_i, d_i, and add edges a_ib_i, b_ic_i, c_id_i, d_ie_i.

Problem 3. Prove that all HHD-free graphs have the DBE property.

Theorem 8 would follow from Theorem 1 of [8] if it were known that substitution preserves the DBE property and that gluing vertices preserves the DBE property. As for the former proposition, it is not known that substitution preserves the DBE property even in the special case where the (not necessarily connected) graph that is being substituted for a vertex has only two vertices.

Problem 4. Prove that splitting a vertex into adjacent twins preserves the DBE property.

Problem 5. Prove that splitting a vertex into nonadjacent twins preserves the DBE property.

Problem 6. Prove that gluing vertices preserves the DBE property.

Ehsan Chiniforooshan and Xiaomin Chen [personal communication] solved a special case of Problem 6:

Theorem 9. *All graphs that can be constructed from graphs of diameter* 2 *by repeatedly gluing vertices have the DBE property.*

Gallai graphs (also known as *i*-triangulated graphs) are a common generalization of chordal graphs and bipartite graphs: every odd cycle of length at least five has at least two non-crossing chords.

Problem 7. Prove that all Gallai graphs have the DBE property.

Since every Gallai graph with no clique-cutset is either a complete multipartite graph or else the join of a connected bipartite graph and a clique [25], Problem 7 is related to proving that

(?) all graphs that can be constructed from graphs of diameter 2 by repeated gluing along cliques have the DBE property,

which would strengthen Theorem 9.

Theorems 8 and 9 highlight the theme of building classes of graphs with the DBE property from prescribed classes by prescribed operations. One of the many additional variations on this theme goes as follows:

Problem 8. Prove that all graphs that can be constructed from bipartite graphs by repeated splitting of vertices into adjacent twins have the DBE property.

The family of *perfect graphs* [14] is, by definition, closed under taking induced subgraphs. Bisplit graphs, HHD-free graphs, and Gallai graphs are subfamilies of this family and they are also closed under taking induced subgraphs. This property seems irrelevant to graph metric; bisplit graphs, HHD-free graphs, and Gallai graphs are featured here just because they have been studied elsewhere and their structure is well understood. The last problem in this subsection concerns a possible strengthening of Conjecture 2 for graphs in another family closed under taking induced subgraphs.

Every bridge in a graph defines a universal line, but not every universal line is defined by a bridge: for instance, the universal line in the wheel with five vertices is defined only by pairs of nonadjacent vertices. This graph and many other examples of bridgeless graphs with universal lines contain an induced subgraph isomorphic to C_4. Yori Zwols [personal communication] conjectured that the answer to the following question is 'true':

Problem 9. True or false? Every C_4-free graph G has at least $|G|$ distinct lines or a bridge.

In March 2018, Martin Matamala and José Zamora [31] proved his conjecture for bipartite graphs (see Theorem 16 in Subsection 3.7).

3.4 A red herring?

The lower bound n in Conjecture 2 is inherited from the more general Conjecture 1. In the more general context of metric spaces, this bound (if at all valid) is tight (consider $n-1$ collinear points in the Euclidean plane and a point off their line). In the more restricted context of graphs, it may be so far from being tight as to be downright misleading:

Conjecture 3. All graphs G without a universal line have

$$\Omega(|G|^{4/3}) \text{ distinct lines.}$$

This conjecture emerged in our ConCoCO discussions of a theorem implying that all graphs G of diameter at most 2 have $\Omega(|G|^{4/3})$ distinct lines [13, Theorem 3]. As noted in [13, Theorem 3], its lower bound is best possible: complete multipartite graphs with $\Theta(|G|^{2/3})$ parts of sizes in $\Theta(|G|^{1/3})$ have $\Theta(|G|^{4/3})$ distinct lines and no universal line. There are many other graphs with these properties: Example 7.8 in [3] exhibits arbitrarily large graphs G with $\Theta(|G|^{4/3})$ distinct lines, no universal line, and unbounded diameter. A neat variation on this theme has been pointed out by Xiaomin Chen: each graph in his class consists of $\Theta(n^{2/3})$ chordless cycles of lengths in $\Theta(n^{1/3})$ that, apart from a vertex common to all of them, are pairwise vertex-disjoint.

Conjecture 3 is known to be valid for graphs of bounded diameter: here is a more general result.

Theorem 10. [3, Theorem 7.4] *If $d(n) = o(n)$, then all graphs with n vertices and diameter $d(n)$ have $\Omega((n/d(n))^{4/3})$ distinct lines.*

Conjecture 3 is also known to be valid for graphs where no line contains another (see Theorem 13 in Subsection 3.5). Its validity would not imply validity of Conjecture 2: it would imply only that counterexamples to Conjecture 2, if any, are finitely many. A plausible common strengthening of both conjectures goes as follows:

Conjecture 4. For every graph G with no universal line there is a complete multipartite graph with no universal line, as many vertices as G, and at most as many distinct lines as G.

To see that Conjecture 4 is indeed a common strengthening of both Conjecture 2 and Conjecture 3, let $f(n)$ denote the smallest number of distinct lines in a complete multipartite graph with n vertices and no universal line. Consider a complete k-partite graph H with n_i vertices in the i-th part and n vertices altogether. If H has no universal line, then $k \geq 3$ and $n_i \neq 2$ for all i, in which case H has $\binom{k}{2} + \sum_{i=1}^{k} \binom{n_i}{2}$ distinct lines. To see that $f(n) \geq n$, assume without loss of generality that $n_i = 1$ when $1 \leq i \leq m$ and $n_i \geq 3$ when $m < i \leq k$ for some m; then observe that

$$\binom{k}{2} + \sum_{i=1}^{k} \binom{n_i}{2} \;=\; \binom{k}{2} + \sum_{i=m+1}^{k} \binom{n_i}{2} \;\geq\; k + \sum_{i=m+1}^{k} n_i \;=\; k + (n-m) \;\geq\; n.$$

To see that $f(n) = \Omega(n^{4/3})$, observe that

$$\binom{k}{2} + \sum_{i=1}^{k} \binom{n_i}{2} \geq \binom{k}{2} + k\binom{n/k}{2} \geq \frac{1}{2}\left(k^2 + \frac{n^2}{k}\right) - n \geq \left(\frac{27}{32}\right)^{1/3} n^{4/3} - n.$$

Conjecture 4 was suggested by recent experimental results of Yori Zwols. He computed the smallest number of distinct lines in graphs with at most 11 vertices and no universal line. With a single exception, graphs attaining the minimum turned out to be complete multipartite. The exception, which is the complement of the Petersen graph, has 15 distinct lines, just like the complete multipartite graphs $K_{3,3,4}$ and $K_{1,3,3,3}$.

3.5 Families of lines

Families of lines in graphs have properties that may seem outlandish to a visitor from a Euclidean space: for instance, the star with vertices $1,2,3,4$ and edges 12, 13, 14 has lines $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{1,2,3,4\}$. Every two lines in a Euclidean space share at most one point, which is not the case in this example.

Problem 10. How difficult is it to recognize hypergraphs whose hyperedge set is the family of lines in some graph?

A graph is *geometric dominant* [12] if none of its lines contains another. In particular, if every two lines in a graph share at most one vertex, then this graph is geometric dominant.

Theorem 11. [12, Theorem 4] *Every two lines in a graph share at most one vertex if and only if this graph is complete or a path or* C_4.

Geometric dominant graphs not specified in Theorem 11 are called *nontrivial.* Nontrivial geometric dominant graphs with n vertices may be hard to find when n is small (the smallest one is the wheel with six vertices), but they are abundant when n is large:

Theorem 12. [12, Theorem 5] *If* $p(n)^3 n/\log n \to \infty$ *and* $(1-p(n))^2 n/\log n \to \infty$ *as* $n \to \infty$*, then the random graph* $\mathscr{G}_{n,p(n)}$ *is almost surely geometric dominant.*

Problem 11. Prove that all geometric dominant graphs have the DBE property.

Theorem 13. [12, Theorem 8] *All nontrivial geometric dominant graphs G have $\Omega(|G|^{4/3})$ distinct lines.*

Proving that all nontrivial geometric dominant graphs have bounded diameter would make Theorem 13 a corollary of Theorem 10. Even a stronger statement might be true:

Problem 12. [12, Question 1] True or false? All nontrivial geometric dominant graphs have diameter 2.

3.6 Equivalence relations

Ceterum autem censeo Carthaginem delendam esse (besides, I also believe that Carthage must be destroyed) was Cato the Elder's stock conclusion to all his speeches in the Roman Senate, irrespective of their topic. With similar persistence, Adrian Bondy liked to point out again and again in our ConCoCO discussions that our progress would get a great boost if we understood which equivalence relations $\equiv$ on the edge sets of K_n arise from graphs with n vertices (or, more generally, from metric spaces on n points) in the sense that $ab \equiv xy \Leftrightarrow L(ab) = L(xy)$.

Section 6 of [3] contains results on distinct pairs of vertices that define the same line. In its notation (Definition 6.2),

$$\begin{aligned} I(a,b) &= \{z:\ z \text{ lies between } a \text{ and } b\}, \\ O(a,b) &= \{z:\ a \text{ lies between } z \text{ and } b \text{ or } b \text{ lies between } a \text{ and } z\} \end{aligned}$$

(so that $L(ab) = \{a,b\} \cup I(a,b) \cup O(a,b)$; in its terminology (Definitions 6.3 – 6.5 and Lemma 6.9), a *parallelogram* is an ordered 4-tuple (a,b,c,d) of distinct vertices such that

- $dist(a,b) = dist(c,d)$,
- $dist(b,c) = dist(d,a)$,
- $dist(a,c) = dist(b,d) = dist(a,b) + dist(b,c)$.

Theorem 14. [3, Lemma 6.6] *Let G be a graph and let e, f be distinct edges of the complete graph on the vertex set of G. If $L_G(e) = L_G(f)$, then the endpoints of e can be labeled a,b and the endpoints of f can be labeled c,d (possibly $b = c$) in such a*

way that
(α) b lies between a and c; c lies between b and d; both b,c lie between a and d or
(β) (a,b,c,d) is a parallelogram and $I(a,b) = I(c,d) = \emptyset$ or
(γ) (a,c,b,d) is a parallelogram and $O(a,b) = O(c,d) = \emptyset$.

Problem 13. How difficult is it to recognize equivalence relations $\equiv$ such that $ab \equiv xy$ if and only if $L_G(ab) = L_G(xy)$ for some graph G?

Some of the candidates $\equiv$ partitioning edge sets of K_n into classes $C_1,\dots C_m$ are rejected by the following procedure.

Algorithm G:
for $i = 1$ to m **do** $L_i =$ the set of all endpoints of edges in C_i **end**
while there are pairwise distinct vertices u,v,w and (not necessarily distinct) subscripts i,j such that $uv \in C_i$, $w \notin L_i$, $vw \in C_j$, $u \in L_j$
do add w to L_i;
end
if there are distinct i,j such that $|L_i| = |L_j| = n$
then **return** message DOES NOT ARISE FROM ANY GRAPH;
else **return** message DON'T KNOW;
end

For instance, given classes $C_1 = \{12,23,34\}$ and $C_2 = \{13,24,14\}$, Algorithm G constructs $L_1 = L_2 = \{1,2,3,4\}$, and so it returns message DOES NOT ARISE FROM ANY GRAPH. Nevertheless, Algorithm G does not eliminate all inputs that do not arise from any graph. For instance, given the partition into classes $C_1 = \{14\}$, $C_2 = \{24\}$, $C_3 = \{34\}$, $C_4 = \{12,23,13\}$ that does not arise from any graph, Algorithm G constructs $L_1 = \{1,4\}$, $L_2 = \{2,4\}$, $L_3 = \{3,4\}$, $L_4 = \{1,2,3\}$, and so it returns message DON'T KNOW.

Correctness of Algorithm G follows from the observation that, for all graphs G such that $L_G(uv) = L_G(xy)$ if and only if uv and xy belong to the same C_i, its **while** loop maintains the invariant $uv \in C_i \;\Rightarrow\; L_G(uv) \supseteq L_i$.

3.7 An interpolation

Let us call an unordered pair uv of vertices a *mighty pair* if $L(uv)$ is universal, let $\lambda(G)$ stand for the number of distinct lines in G, and let $\mu(G)$ stand for the number of mighty pairs in G. In this notation, Conjecture 2 states that

$$\lambda(G) \geq |G| \;\vee\; \mu(G) > 0.$$

A stronger conjecture interpolates between the two operands of the disjunction:

Conjecture 5. [5, Conjecture 2.3] All graphs G satisfy $\lambda(G)+\mu(G)\geq |G|$.

Let us say that a graph G has the *AMRZ property* if $\lambda(G)+\mu(G)\geq |G|$.

Theorem 15. [5, corollary of Theorem 2.1] *All graphs that can be constructed from chordal graphs by repeated substitutions and gluing vertices have the AMRZ property.*

Theorem 2.1 of [5] is stronger than Theorem 15: except for six graphs that have the AMRZ property, it replaces the AMRZ property by the property that the number of lines plus the number of bridges is at least the number of vertices.

In March 2018, Martin Matamala and José Zamora proved that all bipartite graphs have the AMRZ property:

Theorem 16. [31, corollary of Theorem 19] *In all bipartite graphs except for C_4 and $K_{2,3}$, the number of lines plus the number of bridges is at least the number of vertices.*

Theorem 19 of [31] is stronger than Theorem 16: it replaces the 'number of lines' by 'number of lines determined by pairs of vertices at distance 2'.

Conjecture 6. [5, Conjecture 2.2] In all graphs with no pendant edges except for finitely many cases, the number of lines plus the number of bridges is at least the number of vertices.

Here are Problems 3, 4, 5, 6 with 'DBE property' replaced by 'AMRZ property' and phrased more cautiously:

Problem 14. True or false? All HHD-free graphs have the AMRZ property.

Problem 15. True or false? Splitting a vertex into adjacent twins preserves the AMRZ property.

Problem 16. True or false? Splitting a vertex into nonadjacent twins preserves the AMRZ property.

Problem 17. True or false? Gluing vertices preserves the AMRZ property.

4 Beyond graphs

4.1 Metric spaces

Just as all our graphs have at least two vertices, all our metric spaces have at least two points.

In the domain of metric spaces not necessarily arising from graphs, Conjecture 1 has been verified, in addition to its Euclidean case (Theorem 1), in another special case, that of nearly all finite subspaces of $(\mathbf{R}^2, \ell_1)$. Here,'nearly all' means *non-degenerate* in the sense that no two points in the ground set share a coordinate.

Theorem 17. [28, Theorem 1.1] *Every non-degenerate finite subspace of* $(\mathbf{R}^2, \ell_1)$ *has the DBE property.*

Problem 18. Prove Theorem 17 with the non-degeneracy assumption dropped.

Theorem 18. [28, Theorem 1.2] *Every finite subspace of* $(\mathbf{R}^2, \ell_1)$ *has at least* $|V|/37$ *distinct lines or a universal line.*

Problem 18 is a stepping stone toward proving that

(?) every finite subspace of every $(\mathbf{R}^d, \ell_1)$ has the DBE property.

Allowing arbitrary values of d, but restricting the range of vectors in V may seem to create another stepping stone, namely, proving that

(?) every finite subspace of every $(\{0,1\}^d, \ell_1)$ has the DBE property.

However, this restriction does not make the problem any easier: $(\{0,1\ldots,k\}^d, \ell_1)$ is isometrically embeddable in $(\{0,1\}^{kd}, \ell_1)$. To see this, allocate an ordered set of k coordinates to each of the original d coordinates and, within this set, represent value x by 1s in the first x positions followed by 0s in the last $k-x$ positions.

There is nothing special about metric spaces with the ℓ_∞ metric: every metric space $(V, dist)$ is isometrically embeddable in $(\mathbf{R}^{|V|}, \ell_\infty)$. To see this, enumerate the elements of V as $v_1, \ldots, v_m$ and map each v to $(dist(v,v_1), \ldots, dist(v,v_m))$. (A related theorem of Fréchet [23] states that every separable metric space is isometrically embeddable in the space of all bounded sequences of real numbers endowed with the supremum norm.) Since $(\mathbf{R}^2, \ell_2)$ and $(\mathbf{R}^2, \ell_\infty)$ are isometric (one isometry maps (x,y) to $(x+y, x-y)$), asking for a proof that all finite subspaces of $(\mathbf{R}^2, \ell_\infty)$ have the DBE property is just another way of stating Problem 18.

Theorem 5 extends to metric spaces with a weaker lower bound:

Theorem 19. [3, Theorem 3.1] *Every metric space on n points has $\Omega(n^{1/2})$ distinct lines or a universal line.*

Theorem 7 extends to metric spaces:

Theorem 20. [16, Theorem 1] *Every metric space on n points with distances in $\{0,1,2\}$ has the DBE property.*

The notions of λ and μ introduced in Subsection 3.7 extend from graphs to metric spaces. In March 2018, Martin Matamala and José Zamora [31] proved that the 'DBE property' in Theorem 20 can be replaced by 'AMRZ property':

Theorem 21. [31, Theorem 11] *Every metric space on n points ($n \geq 3$) with distances in $\{0,1,2\}$ satisfies $\lambda + \max\{\mu - 1, 0\} \geq n$.*

A special case of Theorem 10 extends to metric spaces:

Theorem 22. [3, Theorem 5.3] *Every metric space on n points with distances in $\{0,1,2,3\}$ has $\Omega(n^{4/3})$ distinct lines.*

It is conceivable that the conclusion of this theorem remains valid even when the hypothesis is relaxed:

Problem 19. Conjecture 1.3 of [3]: Every metric space on n points with a constant number of distinct distances has $\Omega(n^{4/3})$ distinct lines.

This conjecture, if valid, would subsume Theorem 10 with constant $d(n)$. A partial result in its direction goes as follows:

Theorem 23. [3, Theorem 4.3] *Every metric space on n points ($n \geq 2$) with at most k distinct nonzero distances has at least $n/5k$ distinct lines.*

4.2 Pseudometric betweenness

To construct all lines in a prescribed metric space M, we need not know its distance function $dist$. The ternary relation $B(M)$ defined by

$$(u,v,w) \in B(M) \;\Leftrightarrow\; u,v,w \text{ are all distinct and } dist(u,v)+dist(v,w) = dist(u,w)$$

suffices: lines in M are determined by

$$L(xy) \;=\; \{x,y\} \cup \{z:\; (x,y,z) \in B(M) \vee (y,z,x) \in B(M) \vee (z,x,y) \in B(M)\}. \quad (1)$$

A ternary relation is called a *metric betweenness* if it is isomorphic to some $B(M)$. Menger [32] seems to have been the first to study these relations. He pointed out that every metric betweenness B has properties

(M0) $(u,v,w) \in B \;\Rightarrow\; u,v,w$ are three distinct points,
(M1) $(u,v,w) \in B \Rightarrow\; (w,v,u) \in B$,
(M2) $(u,v,w) \in B \Rightarrow\; (u,w,v) \notin B$,
(M3) $(u,v,w),(u,w,x) \in B \;\Rightarrow\; (u,v,x),\; (v,w,x) \in B$.

Following [7], a ternary relation B is called a *pseudometric betweenness* if it has properties (M0), (M1), (M2), (M3). Lines in a pseudometric betweenness B are defined by by (1) with B in place of $B(M)$.

Euclidean betweenness is of course pseudometric and it has additional properties

(M4) $(u,v,w),(v,w,x) \in B \;\Rightarrow\; (u,v,x),\; (u,w,x) \in B$,
(M5) $(u,v,w),(u,v,x) \in B \;\Rightarrow\; (u,w,x),(v,w,x) \in B \vee (u,x,w),(v,x,w) \in B$,
(M6) $(u,v,x),\; (u,w,x) \in B \;\Rightarrow\; (u,v,w),(v,w,x) \in B \vee (u,w,v),(w,v,x) \in B$,

but not every pseudometric betweenness with these properties is Euclidean: for instance,

$$\{(a_1,b_1,c_1),\; (a_1,b_2,c_2),\; (a_2,b_1,c_2),\; (a_2,b_2,c_1),$$
$$(c_1,b_1,a_1),\; (c_2,b_2,a_1),\; (c_2,b_1,a_2),\; (c_1,b_2,a_2)\}$$

is not even metric. (For more on metric betweenness, see [15, Section 6].)

It is conceivable that every pseudometric betweenness has the DBE property. In February 2018, Pierre Aboulker [personal communication] proved this in the special case where the betweenness has a couple of additional properties and suggested that one of these two restrictions may be dropped:

Theorem 24. *Every pseudometric betweenness with properties* (M4) *and* (M5) *has the DBE property.*

Problem 20. True or false? Every pseudometric betweenness with property (M4) has the DBE property.

Theorem 19 extends to pseudometric betweenness, although with an even weaker lower bound:

Theorem 25. [3, Theorem 2.3] *Every pseudometric betweenness on n points has* $\Omega(n^{2/5})$ *distinct lines or a universal line.*

4.3 3*-uniform hypergraphs*

To construct all lines in a prescribed pseudometric betweenness B, we need not know the order of the elements in each triple of B. The set $T(B)$ of unordered triples defined by

$$T(B) = \{\{u,v,w\} : (u,v,w) \in B\}.$$

suffices: lines in B are determined by

$$L(xy) \ = \ \{x,y\} \cup \{z : \ \{x,y,z\} \in T(B)\}. \tag{2}$$

Following [11], *lines in a* 3*-uniform hypergraph* with hyperedge set T are defined by (2) with T in place of $T(B)$. Theorem 1 cannot be generalized from the Euclidean plane all the way to 3-uniform hypergraphs:

Theorem 26. [11, Theorem 3] *There are arbitrarily large* 3*-uniform hypergraphs with n vertices , no universal line, and* $\exp(O(\sqrt{\log n}))$ *distinct lines.*

Nevertheless, the number of distinct lines in 3-uniform hypergraphs with n vertices and no universal line grows beyond every bound as n tends to infinity:

Theorem 27. [2, Theorem 1] *All* 3*-uniform hypergraphs with n vertices have at least* $(2-o(1))\lg n$ *distinct lines or a universal line.*

Here are four classes of 3-uniform hypergraphs that are known to have the DBE property:

Theorem 28. [4, Theorem 3] *If H is a* 3*-uniform hypergraph such that some graph G shares its vertex set with H and three vertices form a hyperedge in H if and only if they are pairwise adjacent in G, then H has the DBE property.*

Theorem 29. [7, Theorems 2, 5, 6] *If, in a* 3*-uniform hypergraph with at least two vertices,*

(a) no four vertices induce two hyperedges or
(b) no four vertices induce one or three hyperedges or
(c) no four vertices induce four hyperedges,

then the hypergraph has the DBE property.

By the way, Theorem 2 of [7] goes beyond the first part of Theorem 29 by describing all 3-uniform hypergraphs where no four vertices induce two hyperedges and the number of distinct lines equals the number of vertices. This theorem is a generalization of the real De Bruijn–Erdős theorem (quoted here as Theorem 3) since every family $\mathscr{L}$ of subsets of a set V such that every two distinct points of V belong to precisely one member of $\mathscr{L}$ is the family of lines in a 3-uniform hypergraph where no four vertices induce two or three hyperedges.

Theorem 29 suggests the following questions:

Problem 21. [7, Question 2] True or false? If, in a 3-uniform hypergraph, every sub-hypergraph induced by four vertices has at least two hyperedges, then the hypergraph has the DBE property.

Problem 22. [7, Question 3] True or false? If, in a 3-uniform hypergraph, every sub-hypergraph induced by four vertices has one or two or four hyperedges, then the hypergraph has the DBE property.

Here is a counterpart of Problem 13 in the context of hypergraphs:

Problem 23. How difficult is it to recognize equivalence relations $\equiv$ such that $ab \equiv xy$ if and only if $L_H(ab) = L_H(xy)$ for some 3-uniform hypergraph H?

The following procedure rejects some of the candidates $\equiv$ partitioning edge sets of K_n into classes $C_1, \ldots C_m$ (just like Algorithm G does), accepts some others (which Algorithm G never does), and gives up in the remaining cases.

Algorithm H:
for $i = 1$ to m **do** $L_i =$ the set of all endpoints of edges in C_i **end**
while there are pairwise distinct vertices u, v, w and (not necessarily distinct) subscripts i, j such that $uv \in C_i$, $w \notin L_i$, $vw \in C_j$, $u \in L_j$
do add w to L_i;
end
if $L_i \neq L_j$ whenever $i \neq j$
then **return** the hypergraph with hyperedge set consisting of all $\{u, v, w\}$ such that $u.v.w$ are pairwise distinct and $uv \in C_i$, $w \in L_i$ for some i;
else **if** there are distinct i, j such that $|L_i| = |L_j| = n$
then **return** message DOES NOT ARISE FROM ANY HYPERGRAPH;
else **return** message DON'T KNOW;
end
end

For instance, given classes $C_1 = \{14\}$, $C_2 = \{24\}$, $C_3 = \{34\}$, $C_4 = \{12, 23, 13\}$, Algorithm H constructs $T = \{\{1, 2, 3\}\}$ and $L_1 = \{1, 4\}$, $L_2 = \{2, 4\}$, $L_3 = \{3, 4\}$, $L_4 = \{1, 2, 3\}$, and so it returns the hypergraph with hyperedge set T. Given classes $C_1 = \{15\}$, $C_2 = \{25\}$, $C_3 = \{35\}$, $C_4 = \{45\}$, $C_5 = \{12, 23, 34\}$, $C_6 = \{13, 24, 14\}$, Algorithm H constructs $L_5 = L_6 = \{1, 2, 3, 4\}, \ldots$, and so it returns message DON'T KNOW.

Correctness of Algorithm H follows from the observation that, for all 3-uniform hypergraphs H such that $H(uv) = L_H(xy)$ if and only if uv and xy belong to the same C_i, its **while** loop maintains the invariant $uv \in C_i \Rightarrow L_H(uv) \supseteq L_i$.

Of course, problems analogous to Problem 23 can be posed also with 'metric spaces' or 'pseudometric betweenness' in place of '3-uniform hypergraphs', but there we have nothing beyond Algorithm G.

4.4 Recognition problems

We have been discussing objects in a hierarchy of four levels:

1. metric spaces arising from graphs,
2. general metric spaces,
3. pseudometric betweenness,
4. 3-uniform hypergraphs.

Because every undirected graph G gives rise to its metric space $M(G)$, every metric space M gives rise to its pseudometric betweenness $B(M)$, and every pseudometric betweenness B gives rise to its 3-uniform hypergraph $H(B)$ with hyperedge set $T(B)$, this four-level hierarchy suggests six questions:

Question 12: Does a prescribed metric space M arise
from a graph G as $M = M(G)$?
Question 13: Does a prescribed pseudometric betweenness B arise
from a graph G as $B = B(M(G))$?
Question 14: Does a prescribed 3-uniform hypergraph H arise
from a graph G as $H = H(B(M(G)))$?
Question 23: Does a prescribed pseudometric betweenness B arise
from a metric space M as $B = B(M)$?
Question 24: Does a prescribed 3-uniform hypergraph H arise
from a metric space M as $H = H(B(M))$?
Question 34: Does a prescribed 3-uniform hypergraph H arise
from a pseudometric betweenness B as $H = H(B)$?

- Question 12 is easy since only one graph G may satisfy $M = M(G)$ with the prescribed metric space M: two vertices are adjacent in G if and only if their distance in M is 1.
- Question 13 is also easy since only one graph G may satisfy $B = B(M(G))$ with the prescribed pseudometric betweenness B: vertices u, w are adjacent in G if and only if no v satisfies $(u, v, w) \in B$.
- Question 14: Let us call a 3-uniform hypergraph H *graphic* if there is a graph G such that $H = H(B(M(G)))$. An induced sub-hypergraph of a graphic hypergraph may not be graphic. One example is the $H(B(M(G)))$ where G consists of the cycle with edges 12, 23, 34, 45, 56, 61, and the additional vertex 7 adjacent to

the antipodal vertices 2 and 5. Here, the sub-hypergraph induced by the four vertices $1,2,3,5$ is not graphic.

Problem 24. How difficult is it to recognize graphic hypergraphs?

- Question 23 can be answered in polynomial time: see [15, Section 6].
- Question 24: Following [7, Section 3], let us call a 3-uniform hypergraph *metric* if there is a metric space M such that $H = H(B(M))$. Not all metric hypergraphs are graphic: one example is the 3-uniform hypergraph with four vertices and one hyperedge. This hypergraph arises from the metric space with

  $$dist(a,b) = dist(b,c) = 1, \quad dist(a,c) = dist(a,d) = dist(b,d) = dist(c,d) = 2$$

  and is not graphic. All induced sub-hypergraphs of a metric hypergraph are metric; a 3-uniform hypergraph is called *minimal non-metric* if it is not metric, but all its proper induced sub-hypergraphs are. Three examples of minimal non-metric hypergraphs are given in [7]. Are there infinitely many minimal non-metric hypergraphs?

Problem 25. How difficult is it to recognize metric hypergraphs?

- Question 34: Following [7, Section 3] again, let us call a 3-uniform hypergraph *pseudometric* if there is a pseudometric betweenness B such that $H = H(B)$. Not all pseudometric hypergraphs are metric: one example is the *Fano hypergraph,* whose hyperedges are the lines of the projective plane of order 2. Like all 3-uniform hypergraphs in which no two hyperedges share two vertices, it is pseudometric; since it does not have the Sylvester–Gallai property, it is not metric [10]. All induced sub-hypergraphs of a pseudometric hypergraph are pseudometric; a 3-uniform hypergraph is called *minimal non-pseudometric* if it is not pseudometric, but all its proper induced sub-hypergraphs are. The three examples of minimal non-metric hypergraphs given in [7] are also minimal non-pseudometric. Are there infinitely many minimal non-pseudometric hypergraphs?

Problem 26. How difficult is it to recognize pseudometric hypergraphs?

Here are counterparts of Problem 10 on the higher levels of the four-level hiearchy:

Problem 27. How difficult is it to recognize hypergraphs whose hyperedge set is the family of lines in some metric space?

Problem 28. How difficult is it to recognize hypergraphs whose hyperedge set is the family of lines in some pseudometric betweenness?

Problem 29. How difficult is it to recognize hypergraphs whose hyperedge set is the family of lines in some 3-uniform hypergraph?

Finally, one could also ask how difficult is it to recognize mappings $f : \binom{V}{2} \to 2^V$ such that some object on a specified level of the hierarchy has $L(uv) = f(uv)$ for all uv. However, this recognition problem is easy on the highest level: the following propositions are logically equivalent.

(A) Some 3-uniform hypergraph H has $L_H(uv) = f(uv)$ for all uv.
(B) If u, v, w are pairwise distinct, then $w \in f(uv) \;\Leftrightarrow\; v \in f(uw) \;\Leftrightarrow\; u \in f(vw)$.

Furthermore, if (B) is satisfied, then the H featured in (A) is unique (its hyperedges are all $\{u, v, w\}$ such that u, v, w are pairwise distinct and $w \in f(uv)$), and so the recognition problem on each of the lower levels is Problem 24 or Problem 25 or Problem 26.

5 Afterword

László Lovász wrote "It is easy to agree that if a conjecture is good, one expects that its resolution should advance our knowledge substantially." [30]. Would resolution of Conjecture 1 advance our knowledge substantially? No. Not unless you stretch the meaning of 'substantially' enough to cover a theme that concerned fewer than two dozen people for the last dozen years. But there is something archetypal about the thrill of taking familiar concepts to unfamiliar territory. Think non-Euclidean geometry. Families of lines in graphs and metric spaces may never find applications comparable to those of hyperbolic geometry, but when we step through the looking glass to study them, what discoveries shall we make? Is Conjecture 2 true or false? Conjecture 4? I want to know.

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Note added in proof: In March 2018, shortly after this paper was submitted for publication, Xiaomin Chen and Ehsan Chiniforooshan solved Problem 1: they proved that all bisplit graphs with at least 80 vertices have the DBE property and verified the rest by computer computations. In addition, they proved that all bisplit graphs with n vertices and no universal line have $\Omega(n^{4/3})$ distinct lines. In August 2018, Laurent Beaudou, Giacomo Kahn, and Matthieu Rosenfeld proved, without the use of a computer, that all bisplit graphs have the DBE property: https://arxiv.org/pdf/1808.08710.pdf

This is a correct [1] version of the paper published by Springer on pages 149–176 of the book *Graph Theory Favorite Conjectures and Open Problems - 2* edited by Ralucca Gera, Teresa W. Haynes, and Stephen T. Hedetniemi.
https://www.springer.com/gp/book/9783319976846

[1] The project manager at SPi Global who handled the production of the book on behalf of Springer neglected to make four corrections requested by the author.